A PROPOSAL FOR A UNIVERSAL ACTUARIAL NOTATION

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ABSTRACT

Notation is at the heart of every mathematical discipline. This paper provides a brief history of actuarial notation, including the development of life notation and attempts to invent a standard non-life notation. We also discuss why actuaries, as early as 1898, have desired a Universal Actuarial Notation and their opinions on what makes such a notation useful. Finally, we submit a proposal for a new actuarial notation.

1 Introduction

Every person who has studied life insurance has encountered its unique notation [1, 2], whereas every person who has studied property and casualty insurance has encountered its inconsistent notation. The subject of notation permeates the history of actuarial science and includes numerous attempts to invent a standard notation for casualty insurance.

Notation was the first topic considered at the Second International Congress of Actuaries in 1898, where they voted on standardizing life notation [3]. The President of the Institute of Actuaries, Thomas Young, motivated the discussions by reflecting upon the history of mathematics. He explained that "progress, or decay, or arrest, of mathematical analysis, had been continually due either to the invention or to the absence of an appropriate symbolism." The Congress unanimously approved the proposed notation, now known as the International Actuarial Notation, but also recognized the need for non-life notation by resolving that "a Universal Notation be adopted, not only for life assurance but for all other branches of assurance."

When the Casualty Actuarial Society formed in 1914, its President, Isaac Rubinow[1] [5], wrote that "the absence of a definite system of symbols places an obstacle in the way of clear writing" [6] and that "a satisfactory nomenclature is a necessary prerequisite for the development of any branch of science" [7]. To fulfill this requirement, one of the first committees established by the CAS was the Committee on Terms, Definitions and Symbols[2] [9]. Their goal was to create a standard casualty actuarial notation; unfortunately, they were unsuccessful.

In 1920, Gustav Michelbacher[3] stressed that "the lack of a simple, comprehensive system of [casualty] notation is becoming extremely embarrassing" and made a "plea for a simple and universal system of notation" [10]. Although the original committee disbanded, "there [was] every indication that the intention was not to drop the subject permanently;" Michelbacher and others called for its prompt re-establishment.

[1]Rubinow was a leading theorist on social insurance whos views influenced President Theodore Roosevelt [4]. The President considered him to be the "greatest single authority upon social security in the United States".

[2]In a footnote in his paper *Liability Loss Reserves* [8], Rubinow expressed both his dissatisfaction with the notation he employed and his hopes for the committee. He stated that he had "no apology to offer for the awkward symbols, except that there is as yet no accepted set of casualty insurance symbols. The appointment of a Committee on Terms, Definitions and Symbols by this Society promises that in the near future the situation will be remedied and there will be no necessity of inventing symbols for every paper."

[3]Michelbacher served as the President of the CAS during 1924-1925 [5]. He made significant contributions to actuarial science which are commemorated by the CAS with an award named after him; The Michelbacher Significant Achievement Award.

A BRIEF HISTORY OF ACTUARIAL NOTATION

1843 – David Jones invents a notation for life insurance in his book *On the Value of Annuities and Reversionary Payments*. [17, 19, 23]

1872 – Jones' notation appears in the *Institute of Actuaries Life Tables* and becomes the official notation of Great Britain. [17, 19]

1877 – George King expands upon the notation in his influential book *The Institute of Actuaries Text-Book, Part II, Life Contingencies*. [17, 24]

1898 – The Second International Congress of Actuaries officially adopts the notation, now known as International Actuarial Notation. [3, 14]

1900 – The Third International Congress approves minor changes to International Actuarial Notation. [14]

1905 – George King and G.J. Lidstone propose notations for pension funds, but no standard is agreed upon. [25, 26]

1912 – John Schooling requests that The Institute of Actuaries standardize sickness notation. [27]

1915 – The CAS establishes a committee to create a casualty actuarial notation. It concludes that the science is too young to permit a stable notation and disbands. [9, 10, 12, 13, 16]

1920 – Sanford Perkins suggests a notation for workers' compensation insurance. CAS members call for the revival of a committee on notation. [10, 12, 28, 29]

1921 – The CAS re-establishes a committee on notation, but it is unable to formulate an acceptable notation. [11]

1934 – Thomas Carlson publishes *Suggestions for a Standard System of Notation in Casualty Actuarial Work*. [13, 30, 31, 32]

1937 – The Eleventh International Congress appoints an international committee to consider refinements to International Actuarial Notation. [14]

1954 – The Fourteenth International Congress updates International Actuarial Notation based on the committee's suggestions. This notation is still in use today. [15, 16]

1968 – Jeffrey Lange proposes linearizing International Actuarial Notation for compatibility with computers, as do many other authors during this time period. [16, 17, 18, 19, 20, 21, 22]

1971 – The report, *A New Actuarial Notation*, is discussed at the Eighth Convention of The Institute of Actuaries of Australia and New Zealand. [33]

1972 – The President of the International Actuarial appoints a sub-committee to investigate linearization and expanding the scope of actuarial notation. [17, 19]

1976 – The Committee on Standard Notation and Nomenclature votes no to linearization and yes to expansion. [19, 34]

Fig. 1: A Brief History of Actuarial Notation

They considered the task "an opportunity for the Society to make a most important contribution to the business of casualty insurance." In response, the CAS established the Committee on Notation in 1921 [11], but it was also unable to create an acceptable notation[4].

In 1934, Thomas Carlson[5] published *Suggestions for a Standard System of Notation in Casualty Actuarial Work* [13]. He bemoaned that "the final spur to my writing this paper ... was a memorandum I received in which the goose-egg familiarly understood to represent zero was used to represent the experience rating off-balance factor, with resulting formulas to make one's head spin." He urged that the time was ripe for the development of a casualty actuarial notation, but a standard notation did not emerge during this time either.

Meanwhile, throughout the 1930s, 1940s, and 1950s, the International Actuarial Notation continued to be refined [14, 15]. In 1954, the Fourteenth International Congress of Actuaries approved the version that is still in use today[6].

During the 1960s and 1970s, many authors invented new notations with a different problem in mind: computers. To be compatible with code, they proposed linearizing actuarial notation [16, 17, 18, 19, 20, 21, 22]. For example, one attempt suggested expressing ${}_{n_1|n_2}a_x^{(2)}$ as $a(x\,;\, n_1 : n_2\,;\, 2)$. In 1972, the President of the International Actuarial Association appointed a sub-committee to investigate the desirability of developing a new actuarial notation [17, 19]. Their investigation focused on two matters: linearization and expansion of scope to other fields such as pensions, disability, and non-life insurance. They voted no to linearization and yes to expansion.

Beyond the 1970s, discussions on notation fade away. The purpose of this paper is to reopen the discussion and propose a Universal Actuarial Notation. Before the proposal, we take a closer look at notations to understand why they are essential and what makes a particular notation useful.

2 A Closer Look at Notations

2.1 Why are Notations Essential?

The most commonly cited reason for the necessity of a standard notation is to improve communication between individuals, committees, and organizations.

Individuals
"[The] lack of a standard notation implies that each author of a technical paper must develop [their] own notation which [their] reader must learn. This is time consuming for both writer and reader, and can make technical papers more difficult to comprehend, thus leading to unnecessary confusion. Tracing a concept through several papers can be particularly troublesome since the same idea may appear in substantially different form in each author's notation. As a result, it is difficult to make comparisons, to recognize parallelisms, and to extend work from one area to another since the variation in notation tends to obscure similarities and impede pattern recognition." [16]
— Jeffrey Lange[7] (1968)

Committees
"When a committee attacks a common problem and each member starts working in [his or her] own way and using [his or her] own symbols, the result is like a meeting of [people] speaking different languages. Until they all adopted the same language it was a rather difficult job to follow from one memorandum to the next." [28]
— A.L. Kirkpatrick[8] (1921)

[4]The comments by Michelbacher and subsequent actions by the CAS were instigated by a paper written by Sanford B. Perkins entitled *A Suggested System of Standard Notation for Actuarial Work in Workmen's Compensation Insurance* [12]. Perkins served as the Chairman on the Committee on Notation [11] and later, during 1926-1927, served as the President of the CAS [5].

[5]Thomas Carlson would later serve as the President of the CAS during 1951-1952 [5].

[6]The final refinements to International Actuarial Notation were scheduled for approval at the International Congress of Actuaries to be held in 1940, but due to World War II the Congress did not convene [14].

[7]Lange served the CAS as the General Chairman of the Education and Examination Committee during 1977-1979 [5].

[8]Kirkpatrick was a member of the CAS Committee on Notation [11].

Organizations
"A logical system of symbols ... will facilitate friendly intercourse and interchange of ideas between all the actuarial organizations." [6]
— Isaac Rubinow (1915)

As alluded by Lange, notation does more than facilitate communication; it can promote or hinder the progress of an entire field. The issue of notation is not exclusive to actuarial science; the history of calculus exemplifies a notation's ability to advance scientific studies [35]. Isaac Newton and Gottfried Leibniz independently discovered calculus using very different notations; Newton formulated the notation of fluxions, whereas Leibniz invented the differential notation. The two scientists feuded over credit for calculus resulting in a century-long rift between British and non-British mathematicians, who used the notation of fluxions and differentials, respectively.

Thomas Young summarized the rift's effect at the Second International Congress of Actuaries [3]. He said that "in England there had been no more significant feature in the history of British mathematics, than the fact that, after the death of Sir Isaac Newton, the course of mathematical analysis was retarded for nearly 100 years, simply by reason of the circumstances that British mathematicians adhered exclusively to the geometrical methods of Newton, instead of adopting the more flexible and potent instrument which lay in their power in connection with the differential calculus."

Many notable scientists throughout history have also echoed the value of notation.

"It is worth noting that the notation facilitates discovery. This, in a most wonderful way, reduces the mind's labour." [36]
— Gottfried Leibniz (1646 - 1716)

"By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems." [38]
— Alfred North Whitehead (1861 - 1947)

"Such is the advantage of a well constructed language that its simplified notation often becomes the source of profound theories." [37]
— Pierre-Simon Laplace (1749 - 1827)

"We could of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations." [39]
— Richard Feynman (1918 - 1988)

It is clear that notation is essential, but what makes a notation useful? In particular, what criteria must be satisfied for a Universal Actuarial Notation?

2.2 Criteria for a Universal Actuarial Notation

Creating a useful notation is not an exact science. If it were, actuarial science would already have its universal notation. Discussions stimulated by prior attempts at standardization have included opinions on the criteria for a useful notation [10, 28, 29, 30, 31, 32, 40, 41, 42]. Their criteria generally fall into one of the following five categories.

1. Universal
2. Simple
3. Self-Consistent
4. Compatible with Existing Conventions
5. Easy to Produce Typographically

It is almost impossible to satisfy all these criteria simultaneously. A notation must weigh the relative importance of individual criteria in instances when they conflict and explain the reason for choosing one criterion over another.

Universal
For an actuarial notation to be universal, it must be general in that it applies to all sub-fields of insurance[9].

[9] Michelbacher's major criticism of Perkins' suggested notation was that it only applied to workers' compensation insurance [10]. He said that "I would not have one notation for [workers'] compensation insurance and another for accident and health insurance if I could help it."

Only concepts that are common across all specialties can be assigned symbols; there can be no direct reference to the features of one area which do not apply to another. However, this is not to say such features would never appear; otherwise, the notation would be useless.

The most successful notations are both universal and general; consider Newtonian mechanics as an example. The symbol F represents force under all circumstances, whether it be gravitational, centripetal, or elastic. When a problem involves a particular type of force, its specific formula gets assigned to the symbol F; each type of force does not have a dedicated symbol. Without this feature, Newton's second law, $F = ma$, would lose its power and beauty and fragment into a multitude of formulas, obscuring its true meaning.

Simple
A simple notation is both concise and recognizable. These conditions often work in opposition and must be balanced. Kirkpatrick's opinion is that "it is more important to have symbols which are easily recognized and remembered than to have short ones" [28]. Simplicity is difficult to pinpoint because oversimplification can lead to complexity.

When Carlson invented his casualty notation, he strived for simplicity by differentiating similar objects by use of the right subscript and only the right subscript [13]. The abandonment of the left subscript and superscripts resulted in a complicated notation because implementing numerous distinct actuarial ideas into a single index is impractical[10]. As Albert Einstein said, "everything should be made as simple as possible, but no simpler."

Self-Consistent
There are three requirements for notational self-consistency.

1. A symbol cannot represent multiple concepts.
2. A concept cannot be represented by multiple symbols.
3. There can be no contradictory conventions.

There are two crucial caveats to these conditions.

The first caveat is that a sub-symbol of a larger symbol may have different meanings depending on its position. For example, the sub-symbol "3" has entirely different meanings in the symbols 0.3 and 0.03. Notations with this property are known as positional notations[11].

The second caveat is that notational systems can represent individual concepts in two forms: full-form or short-form. The use of short-forms is to encourage simplicity in two contexts.

1. Common usages.
2. Unambiguously implied parameters.

For example, the life notation for the probability of death of a person aged x before reaching the age $x + t$, ${}_tq_x$, is commonly used when $t = 1$ and shortened from ${}_1q_x$ to q_x. Similarly, when the interest rate, i, is unambiguously specified, the notation for an annuity-due with n payments of $1 can be shortened from $\ddot{a}_{\overline{n}|i}$ to $\ddot{a}_{\overline{n}|}$.

Compatible with Existing Conventions
Compatibility is important because communities are accustomed to their existing conventions, and it allows for existing mathematical machinery to be built upon with minimal explanation. As noted in the paper *An Analysis of Mathematical Notations: For Better or For Worse* [42], "the math and science community is quite resilient to changes. Sometimes supporters of rival notations both have a good argument, while sometimes [their preferences are] simply a matter of habit." Significant deviations from established conventions may hurt a notation's chances of adoption by the community.

[10]The complexity of Carlson's notation led him to invent an entirely different system of notation in one of his later papers [16, 43].

[11]Laplace commented on the power of positional notations: "It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity." [44]

Easy to Produce Typographically
Typography is a practical constraint; other practitioners must be able to reproduce the notation to use it. Improved mathematical typesetting systems, such as LaTeX, allow for the creation of custom packages to reproduce a notation that anyone can download and use, for example, the package `actuarialsymbol`. As such, this is not as great a concern as it would have been in the past.

These criteria are considered carefully in the following proposal.

3 Outline Of the Proposal

We begin by borrowing and expanding on an idea from Thomas Carlson [13]. He created his notational system by classifying actuarial terms into two groups: basic terms and delimiting terms. Basic terms stand by themselves, such as claims or premiums, whereas delimiting terms must qualify some basic term, for example, paid or earned. Carlson symbolically distinguished between these two groups by making delimiting terms a subscript of basic terms.

This proposal extends his idea by removing the restriction on the location of delimiting terms and classifying basic terms into three groups.

Auxiliary Basic Terms
Common parameters or factors such as time, discount factors, trends, and off-balance factors.

Primary Basic Terms
Absolute quantities such as claims, premiums, number of claims, and units of exposure.

Secondary Basic Terms
Ratio quantities such as claims-ratio, frequency, and severity.

Many basic terms are random variables whose classification depends on various characteristics, such as a claims distribution. Randomness and classifications are fundamental traits of actuarial science, and so we begin by discussing the notations for random variables and vectors.

4 Random Variables

The notation for random variables is well established within the mathematical community; an upper case letter represents a random variable, and the corresponding lower case letter represents its realizations.

Adhering to this convention creates an obstacle to achieving self-consistency because it halves the number of available symbols[12]. For example, we may wish to choose the convention that the symbols P and p represent premium and probability, respectively; this would be a contradictory convention. The situation is further complicated if we want to use the symbol ρ for another concept since its capitalization is also P.

Phelim Boyle faced a similar issue in his paper *Rate of Return as a Random Variable* [45] and chose to "not follow the conventional statistical practice ... as this would conflict with conventional actuarial notation," namely a and A. Boyle decided that self-consistency is more important than compatibility with existing conventions, and so do we.

Instead of the standard convention, we propose to represent random variables by boldfaced[13] characters and realizations, or constants in general, by regular typeface characters, for example, $\boldsymbol{C}$ and C.

[12] A similar issue arose in 1905 when George King and G. J. Lidstone discussed notations for pension funds [25, 26]. King commented on Lidstone's notation saying that he "proposes to use six good symbols where I use only one, and with our short alphabet we cannot afford such extravagance. It is difficult enough now to find a sufficiency of symbols, and when the investigation of Pension Fund problems has further developed, the difficulty will be increased. Therefore, we must not be prodigal."

[13] In this paper, a variable is bolded using the command `pmb`, which stands for poor man's bold. This command creates a bold effect by superimposing the character on top of itself two times in slightly different positions. One interpretation of this effect is that the location of the true character is uncertain and hence represents a random variable.

An essential aspect of random variables in actuarial science is their layers; they are vital when considering deductibles, limits, and reinsurance. Layers deserve an appropriate notation.

4.1 A Notation for Layers

We define the following notation[14] to represent the layer of $\boldsymbol{C}$ between the lower limit l_0 and upper limit l_1.

$$\langle l_0|\boldsymbol{C}|l_1\rangle = \begin{cases} 0 & -\infty < \boldsymbol{C} < l_0 \\ \boldsymbol{C} - l_0 & l_0 \leq \boldsymbol{C} < l_1 \\ l_1 - l_0 & l_1 < \boldsymbol{C} < \infty \end{cases}$$

We choose this symbolism because it resembles the mathematical operation it represents: the symbol of the random variable is between the symbols of the limits indicating the layer itself, and the angled brackets resemble inequality signs that are a feature of the definition of a layer.

The notation can also be visualized by considering the overlap between the intervals $(-\infty, \boldsymbol{C})$ and (l_0, l_1) for different values of $\boldsymbol{C}$, as shown.

$$\langle l_0|\boldsymbol{C}|l_1\rangle = 0$$

$$\langle l_0|\boldsymbol{C}|l_1\rangle = \boldsymbol{C} - l_0$$

$$\langle l_0|\boldsymbol{C}|l_1\rangle = l_1 - l_0$$

Fig. 2: Visualization of the Layers of a Random Variable and the Corresponding Notation

The expected value of a layer is expressed as $\mathrm{E}\left[\langle l_0|\boldsymbol{C}|l_1\rangle\right]$.

The notation also reveals the additive property of layers in a simplified form.

$$\langle l_0|\boldsymbol{C}|l_2\rangle = \langle l_0|\boldsymbol{C}|l_1\rangle + \langle l_1|\boldsymbol{C}|l_2\rangle \qquad \text{and} \qquad \langle l_0|\boldsymbol{C}|l_L\rangle = \sum_{k=1}^{L}\langle l_{k-1}|\boldsymbol{C}|l_k\rangle$$

We also introduce two short forms for situations when only one of the limits exists. In both cases, the infinite limit may be omitted from the notation as follows.

$$\langle l_0|\boldsymbol{C}\rangle = \langle l_0|\boldsymbol{C}|\infty\rangle \qquad \text{and} \qquad \langle\boldsymbol{C}|l_1\rangle = \langle -\infty|\boldsymbol{C}|l_1\rangle$$

5 Vectors

There are two conventional notations for vectors; they are either represented by boldfaced characters or using an arrow as an accent. To avoid contradictory conventions, we will use the accented arrow, $\vec{k}$.

To distinguish the components of a vector and an index representing the object/observation, we borrow the notation described in *The Hundred-Page Machine Learning Book* [47] and use the right subscript to indicate the object/observation and the right superscript to represent the vector component. For example $\vec{k}_j = \left(k_j^{(1)}, k_j^{(2)}, \ldots, k_j^{(G)}\right)$ where $k_j^{(g)}$ represents the g^{th} component of the vector associated with the object/observation labelled by j.

[14] This notation is inspired by Paul Dirac's bra-ket notation for quantum mechanics [46].

6 Auxiliary Basic Terms

Auxiliary basic terms are standard parameters or factors. For each term, we choose the symbol to align with current conventions [2, 48, 49, 50]; if no convention exists, we choose a symbol that represents its meaning. For example, we choose ϕ to represent the off-balance factor since its meaning is that the difference between two expressions is non-zero[15].

Interest Rate	i	Discount Factor	v	Trend	τ
Off-Balance Factor	ϕ	Credibility	Z	Probability	p, q
Base Rate	β	Characteristic Vector	$\vec{k}$	Base Vector	$\vec{b}$
Layer	l	Apportionment Ratio	a	Year	Y
Absolute Time	T	Relative Time	t		

The difference between absolute and relative times is discussed in the delimiting symbols section.

7 Primary Basic Terms

There are three sets of primary basic terms.

1. Fundamental Objects
2. Insurance Terms
3. Accounting Terms

7.1 Fundamental Objects

The fundamental objects of insurance are the exposure, policy, and book of policies; they are the subject of all insurance equations and are deserving of dedicated symbols.

Exposure	X
Policy	Λ
Book	B

The symbols for exposure and policy serve an additional purpose; they represent the number of units.

Number of exposure units of policy j	X_j
Number of exposure units of book B	X_B
Number of policies in book B	Λ_B

These definitions imply the following relationship.

$$X_B = \sum_{j \in B} X_j \tag{1}$$

7.2 Insurance Terms

The following are the basic insurance terms and the symbols chosen to represent them.

Claims	C	Underwriting Expenses	U	Claim Count	N
Premium	P	Investment Income	I	Reserves	V
Recoveries	R	Profit	Π	Margin	M

These can be used to write the fundamental insurance equation.

$$P = C + U - R - I + \Pi \tag{2}$$

To satisfy the universality criterion, claims are defined to be payments made by an insurer to an insured, beneficiary, or third party on behalf of an insured.

[15] We hope that Thomas Carlson would approve.

7.3 Accounting Terms

The following are the basic accounting terms and the symbols chosen to represent them.

Assets	A	Equity	E
Liabilities	L	Capital	K

These can be used to write the fundamental accounting equation.

$$A = L + E \tag{3}$$

8 Delimiting Terms

We now demonstrate the delimiting terms using claims as a base. Most delimiting terms apply to all basic terms; however, when there are properties that apply solely to claims, they are pointed out. Afterwards, we discuss the unique properties of other basic terms.

The following are the actuarial concepts that are encoded into the notation using delimiters.

Concept	**Description**
Object	Relating to a policy or book
Period	The time interval of consideration
Time-Value	The time-value at a valuation-time given a discount rate
Coverage	A particular product or coverage
Average	Per unit of exposure
Pure vs. Expense	Pure costs vs expense costs
Allocated vs. Unallocated	Costs that can and cannot be assigned to policies
Estimator	Indicates if a quantity is an estimator
Period Basis	Grouping based on types of periods, such as calendar year or policy year
Type	Categories, such as paid, unpaid, reported, or unreported
Level	Restating past quantities to represent future conditions, for example trending

For each delimiter, the following are discussed when applicable.

1. Definition
2. Examples
3. Relationships
4. Short-Forms
5. Extensions

8.1 Object

Definition	The Object Delimiter
Description	Indicates whether a basic symbol refers to a policy, book, or individual payment
Position	Right subscript
Convention	A lower case letter represents a policy, usually j An upper case letter represents a book, usually B A letter followed by a colon and another lower case letter specifies the individual payment, usually $j{:}n$ or $B{:}n$
Mnemonic	Policies are lower level objects and represented by lower case letters Books are higher level objects and represented by upper case letters

Examples

$\boldsymbol{C}_j$	Total claim cost for policy j	$\boldsymbol{C}_B$	Total claim cost for book B
$\boldsymbol{C}_{j:n}$	Cost of n^{th} claim for policy j	$\boldsymbol{C}_{B:n}$	Cost of n^{th} claim for book B
$\boldsymbol{T}_j$	Average claim payment time of policy j	$\boldsymbol{T}_B$	Average claim payment time of book B
$\boldsymbol{T}_{j:n}$	Payment time of n^{th} claim for policy j	$\boldsymbol{T}_{B:n}$	Payment time of n^{th} claim for book B
$\boldsymbol{N}_j$	Total claim count for policy j	$\boldsymbol{N}_B$	Total claim count for book B

Relationships
The equations relating the total claim costs and individual claim costs, (4), average claim time and individual claim time, (5), and book claim count to policy claim count, (6), are

$$\boldsymbol{C}_j = \sum_{n=1}^{\boldsymbol{N}_j} \boldsymbol{C}_{j:n} \qquad \boldsymbol{C}_B = \sum_{n=1}^{\boldsymbol{N}_B} \boldsymbol{C}_{B:n} \tag{4}$$

$$\boldsymbol{T}_j = \frac{1}{\boldsymbol{C}_j} \sum_{n=1}^{\boldsymbol{N_j}} \boldsymbol{C}_{j:n} \cdot \boldsymbol{T}_{j:n} \qquad \boldsymbol{T}_B = \frac{1}{\boldsymbol{C}_B} \sum_{n=1}^{\boldsymbol{N_B}} \boldsymbol{C}_{B:n} \cdot \boldsymbol{T}_{B:n} \tag{5}$$

$$\boldsymbol{N}_B = \sum_{j \in B} \boldsymbol{N}_j \tag{6}$$

Short-Forms
When an equation holds for both policies and books, the object delimiter may be omitted to reveal the general relationship. Using this short-form, equations (4) and (5) become

$$\boldsymbol{C} = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n} \qquad \text{and} \qquad \boldsymbol{T} = \frac{1}{\boldsymbol{C}} \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n} \cdot \boldsymbol{T}_{:n}.$$

Extensions
This notation can be extended in two ways.

1. Continuous Claims
The notation $\boldsymbol{C}_{j:T}$ is used to represent the claim costs at time T for a policy j. The total claim costs and average time relationships become

$$\boldsymbol{C}_j = \int_{-\infty}^{\infty} \boldsymbol{C}_{j:T} dT \qquad \text{and} \qquad \boldsymbol{T}_j = \frac{1}{\boldsymbol{C}_j} \int_{-\infty}^{\infty} \boldsymbol{C}_{j:T} T dT.$$

Replacing j with B in the above provides similar expressions for a book B. The short-form can also be used.

2. Complicated Scenarios
Individual claims could consist of multiple payments. To accommodate such situations, the h^{th} payment of the n^{th} claim of policy j may be written as $\boldsymbol{C}_{j:n:h}$ at time $\boldsymbol{T}_{j:n:h}$ and the individual claim quantities become

$$\boldsymbol{C}_{j:n} = \sum_{h=1}^{\boldsymbol{H}_{j:n}} \boldsymbol{C}_{j:n:h} \qquad \text{and} \qquad \boldsymbol{T}_{j:n} = \frac{1}{\boldsymbol{C}_{j:n}} \sum_{h=1}^{\boldsymbol{H}_{j:n}} \boldsymbol{C}_{j:n:h} \cdot \boldsymbol{T}_{j:n:h}.$$

8.2 Period

Definition	The Period Delimiter
Description	Considers payments from a specific time interval
Position	Right adjacent
Convention	Two times in parentheses

Example
$\boldsymbol{C}(T_1, T_2)$ Cost of claims between times T_1 and T_2, $T_1 \leq T < T_2$

Relationships
The equations relating the costs two adjacent periods, (7), and a period to the total claim costs, (8), are

$$\boldsymbol{C}(T_1, T_3) = \boldsymbol{C}(T_1, T_2) + \boldsymbol{C}(T_2, T_3) \qquad \text{and} \tag{7}$$

$$\boldsymbol{C}(T_1, T_2) = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n} \mathbb{1}(T_1 \leq \boldsymbol{T}_{:n} < T_2) \qquad \text{where} \qquad \mathbb{1}(T_1 \leq T < T_2) = \begin{cases} 1 & T_1 \leq T < T_2 \\ 0 & \text{Otherwise} \end{cases}. \tag{8}$$

Short-Forms
There are two short-forms to consider.

1. Period Beginning at Earliest Possible Claim Time
When a period begins at earliest possible claim time, say T_0, it may be omitted from the notation to become

$$\boldsymbol{C}(T) = \boldsymbol{C}(T_0, T).$$

2. Relative Time
It is often useful to consider times relative to the earliest possible claim time. To denote this we use t to indicate relative time. For example, if $T_1 = T_0 + t_1$ and $T_2 = T_0 + t_2$ we may write

$$\boldsymbol{C}(t_1, t_2) = \boldsymbol{C}(T_0 + t_1, T_0 + t_2) \qquad \text{and} \qquad \boldsymbol{C}(t) = \boldsymbol{C}(T_0 + t) = \boldsymbol{C}(T_0, T_0 + t).$$

8.3 Time-Value

Definition	The Time-Value Delimiter
Description	Evaluates the time-value of a cash flow at a valuation-time T_V with discount rate $v = (1+i)^{-1}$
Position	Right adjacent (after the period delimiter)
Convention	The @ symbol followed by the valuation-time and discount factor in parentheses
Mnemonic	Verbalized as "the time-value of ... at ... "

Example

$\boldsymbol{C}@(T_V, v)$	The time-value of the total claim costs at the valuation-time T_V using discount rate v

Relationships
The time-value is written in terms of the individual claim amounts and times as follows.

$$\boldsymbol{C}@(T_V, v) = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n} \cdot v^{(\boldsymbol{T}_{:n} - T_V)}$$

Short-Forms
There are three short-forms to consider.

1. The time-value of the n^{th} claim
A natural short-form for the time-value of the n^{th} claim is

$$\boldsymbol{C}_{:n}@(T_V, v) = \boldsymbol{C}_{:n} \cdot v^{(\boldsymbol{T}_{:n} - T_V)} \qquad \text{leading to} \qquad \boldsymbol{C}@(T_V, v) = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n}@(T_V, v).$$

2. The discount rate is unambiguously known
Under this scenario the discount rate is omitted from the notation and becomes

$$\boldsymbol{C}@T_V = \boldsymbol{C}@(T_v, v) \qquad \text{leading to} \qquad \boldsymbol{C}@T_V = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n}@T_V.$$

3. The discount rate is unambiguously known and the valuation-time is the present
If the valuation-time is the present, $T_V = T_P$, it can also be omitted and becomes

$$\boldsymbol{C}@ = \boldsymbol{C}@T_P \qquad \text{leading to} \qquad \boldsymbol{C}@ = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n}@.$$

Extensions
This notation can be extended in three ways.

1. Continuous Discounting
In this case, the time-value is written as

$$\boldsymbol{C}@(T_V, v) = \int_{-\infty}^{\infty} \boldsymbol{C}_{:T} \cdot v^{(\boldsymbol{T}_{:n} - T_V)} dT.$$

2. The Expected Present Value
When a cash flow is a random variable, its expected present value is written as

$$\mathrm{E}\left[\boldsymbol{C}@\right] = \sum_{n=1}^{\boldsymbol{N}} \mathrm{E}\left[\boldsymbol{C}_{:n}@\right].$$

3. Non-Constant Discount Factors
If the discount factor is a random variable, this is expressed as $\boldsymbol{C}@(T_V, \boldsymbol{v})$.
If the discount factor is a function of time, this is expressed as $\boldsymbol{C}@\left(T_V, v(T)\right)$.
If the discount factor is both a random variable and a function of time, this is expressed as $\boldsymbol{C}@\left(T_V, \boldsymbol{v}(T)\right)$.

4. Life Notation
This notation unifies and generalizes much of the life notation. Each standard life symbol reduces to a specification of $\boldsymbol{N}_j$, $\boldsymbol{C}_{j:n}$, $\boldsymbol{T}_{j:n}$ and T_V which are used in

$$\boldsymbol{C}_j@(T_V, v) = \sum_{n=1}^{\boldsymbol{N}_j} \boldsymbol{C}_{j:n} \cdot v^{(\boldsymbol{T}_{j:n} - T_V)}.$$

For example, consider the following life symbols. Note that the standard notation uses n for the period length, but to facilitate comparisons this is replaced with N.

Life Notation	T_V	$\boldsymbol{N}_j$	$\boldsymbol{T}_{j:n}$	$\boldsymbol{C}_{j:n}$
$a_{\overline{N}\rceil} = \sum_{n=1}^{N} v^n$	0	N	n	1
$\ddot{a}_{\overline{N}\rceil} = \sum_{n=1}^{N-1} v^n$	0	N	$n-1$	1
$s_{\overline{N}\rceil} = \sum_{n=1}^{N} (1+i)^{n-1}$	N	N	n	1
$\ddot{s}_{\overline{N}\rceil} = \sum_{n=1}^{N} (1+i)^{n-1}$	N	N	$n-1$	1
${}_k\vert a_{\overline{N}\rceil} = \sum_{n=1}^{N} v^{k+n}$	0	N	$n+k$	1
$(Ia)_{\overline{N}\rceil} = \sum_{n=1}^{N} n v^n$	0	N	n	n

Fig. 3: Standard Life Actuarial Notation in Terms of the Proposed Notation

The notation also encompasses continuous payments and random variable cash flows.

$$\bar{a}_{\overline{N}\rceil} = \int_0^N v^T dT \quad \text{corresponds to} \quad \boldsymbol{C}@ \quad \text{where} \quad \boldsymbol{C}_{j:T} = \begin{cases} 1 & 0 \leq T \leq N \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} A_x &= \mathrm{E}\left[v^{\boldsymbol{K}_x + 1}\right] \quad \text{corresponds to} \quad \mathrm{E}\left[\boldsymbol{C}@\right] \quad \text{where} \quad & \boldsymbol{N}_j &= 1 \\ &= \sum_{k=0}^{\infty} v^{k+1} \ {}_k\vert q_x & \boldsymbol{T}_{j:1} &= \boldsymbol{K}_x + 1 \\ & & \boldsymbol{C}_{j:1} &= 1. \end{aligned}$$

In the above, $\boldsymbol{K}_x$ is the curtate future lifetime random variable.

Using the symbol C in this manner is analogous to F from Newtonian mechanics; C represents claims under all circumstances. When a problem involves a particular payment scheme, its specific formula gets assigned to the symbol C rather than having a dedicated symbol.

8.4 Coverage

Definition	The Coverage Delimiter
Description	Specifies different products or coverages
Position	Right superscript
Convention	An index in parentheses representing the coverage/product
Mnemonic	The coverages can be thought to form a vector

Example

$\boldsymbol{C}^{(g)}$ Total claim costs for g^{th} coverage

Relationships

The total claim cost is the sum of the claim costs of each coverage.

$$\boldsymbol{C} = \sum_{g=1}^{G} \boldsymbol{C}^{(g)}$$

8.5 Component

The component delimiter is composed of three parts.

1. Average
2. Total Costs vs. Pure Costs vs. Expense Costs
3. Total Costs vs. Allocated Costs vs. Unallocated Costs

8.5.1 Average

Definition	The Average Delimiter
Description	Average with respect to units of exposure
Position	Accent
Convention	A bar signifies an average cost per unit of exposure No bar signifies total cost
Mnemonic	A bar typically represents average

Example

$\overline{\boldsymbol{C}}$ Average cost with respect to units of exposure

Relationships

The equation relating the total claim costs to average costs with respect to units of exposure is

$$\overline{\boldsymbol{C}} = \frac{\boldsymbol{C}}{X}.$$

8.5.2 Total Costs vs. Pure Costs vs. Expense Costs

Definition	The Pure/Expense Delimiter
Description	A pure cost is a direct payment and expenses are costs associated with pure costs
Position	Accent
Convention	A single dot signifies a pure cost A double dot signifies an expense cost No dot signifies total cost
Mnemonic	Pure costs the primary concern and represented by a single dot Expense costs are the secondary concern and represented by a double dot Expenses are secondary because they exist as a result of pure costs

Examples

$\dot{\boldsymbol{C}}$ Total pure costs

$\ddot{\boldsymbol{C}}$ Total expense costs

Relationships

The equations relating the total claim costs to pure and expense costs, total pure costs to individual pure costs, and total expense costs to individual expense costs are as follows.

$$\boldsymbol{C} = \dot{\boldsymbol{C}} + \ddot{\boldsymbol{C}} \qquad \dot{\boldsymbol{C}} = \sum_{n=1}^{\boldsymbol{N}} \dot{\boldsymbol{C}}_{:n} \qquad \ddot{\boldsymbol{C}} = \sum_{n=1}^{\boldsymbol{N}} \ddot{\boldsymbol{C}}_{:n}$$

8.5.3 Total Costs vs. Allocated Costs vs. Unallocated Costs

Definition The Allocated/Unallocated Delimiter

- Description: An allocated cost can be directly attributed to a policy whereas an unallocated cost cannot.
- Position: Accent
- Convention: A hat signifies an allocated cost
  A check signifies an unallocated cost
  No hat or check signifies total cost
- Mnemonic: A hat resembles the symbol for a policy, Λ, and serves as a reminder that these costs can be allocated to a policy
  A check is the opposite

Example

- $\widehat{\boldsymbol{C}}$ Total allocated costs
- $\widecheck{\boldsymbol{C}}$ Total unallocated costs

Relationships

The equation relating the total claim costs to total allocated and unallocated costs is

$$\boldsymbol{C} = \widehat{\boldsymbol{C}} + \widecheck{\boldsymbol{C}}. \tag{9}$$

By definition all policy-object costs must be allocated, so all policy-object unallocated (checked) variables are defined to be zero.

$$\widecheck{\boldsymbol{C}}_j = 0 \qquad \text{implying} \qquad \boldsymbol{C}_j = \widehat{\boldsymbol{C}}_j.$$

8.5.4 The Component Delimiter

Definition The Component Delimiter

- Description: The combination of the average, pure/expense, and allocated/unallocated delimiters
- Position: Accent

Examples

- $\widehat{\dot{\boldsymbol{C}}}$ Allocated pure costs
- $\widecheck{\dot{\boldsymbol{C}}}$ Unallocated pure costs
- $\widehat{\ddot{\boldsymbol{C}}}$ Allocated expense costs
- $\widecheck{\ddot{\boldsymbol{C}}}$ Unallocated expense costs
- $\overline{\widehat{\boldsymbol{C}}}$ Average allocated costs
- $\overline{\widecheck{\boldsymbol{C}}}$ Average unallocated costs
- $\overline{\dot{\boldsymbol{C}}}$ Average pure costs
- $\overline{\ddot{\boldsymbol{C}}}$ Average expense costs
- $\overline{\widehat{\dot{\boldsymbol{C}}}}$ Average allocated pure costs
- $\overline{\widecheck{\dot{\boldsymbol{C}}}}$ Average unallocated pure costs
- $\overline{\widehat{\ddot{\boldsymbol{C}}}}$ Average allocated expense costs
- $\overline{\widecheck{\ddot{\boldsymbol{C}}}}$ Average unallocated expense costs

Relationships

The equations relating various sub-components to one another are as follows.

$$\dot{\boldsymbol{C}} = \widehat{\dot{\boldsymbol{C}}} + \widecheck{\dot{\boldsymbol{C}}} \qquad \ddot{\boldsymbol{C}} = \widehat{\ddot{\boldsymbol{C}}} + \widecheck{\ddot{\boldsymbol{C}}} \qquad \widehat{\boldsymbol{C}} = \widehat{\dot{\boldsymbol{C}}} + \widehat{\ddot{\boldsymbol{C}}} \qquad \widecheck{\boldsymbol{C}} = \widecheck{\dot{\boldsymbol{C}}} + \widecheck{\ddot{\boldsymbol{C}}}$$

$$\overline{\dot{\boldsymbol{C}}} = \overline{\widehat{\dot{\boldsymbol{C}}}} + \overline{\widecheck{\dot{\boldsymbol{C}}}} \qquad \overline{\ddot{\boldsymbol{C}}} = \overline{\widehat{\ddot{\boldsymbol{C}}}} + \overline{\widecheck{\ddot{\boldsymbol{C}}}} \qquad \overline{\widehat{\boldsymbol{C}}} = \overline{\widehat{\dot{\boldsymbol{C}}}} + \overline{\widehat{\ddot{\boldsymbol{C}}}} \qquad \overline{\widecheck{\boldsymbol{C}}} = \overline{\widecheck{\dot{\boldsymbol{C}}}} + \overline{\widecheck{\ddot{\boldsymbol{C}}}}$$

$$\begin{aligned}\boldsymbol{C} &= \dot{\boldsymbol{C}} + \ddot{\boldsymbol{C}} \\ &= \widehat{\dot{\boldsymbol{C}}} + \widecheck{\dot{\boldsymbol{C}}} + \widehat{\ddot{\boldsymbol{C}}} + \widecheck{\ddot{\boldsymbol{C}}} \\ &= \widehat{\dot{\boldsymbol{C}}} + \widehat{\ddot{\boldsymbol{C}}} + \widecheck{\dot{\boldsymbol{C}}} + \widecheck{\ddot{\boldsymbol{C}}} \\ &= \widehat{\boldsymbol{C}} + \widecheck{\boldsymbol{C}}\end{aligned} \qquad \begin{aligned}\overline{\boldsymbol{C}} &= \overline{\dot{\boldsymbol{C}}} + \overline{\ddot{\boldsymbol{C}}} \\ &= \overline{\widehat{\dot{\boldsymbol{C}}}} + \overline{\widecheck{\dot{\boldsymbol{C}}}} + \overline{\widehat{\ddot{\boldsymbol{C}}}} + \overline{\widecheck{\ddot{\boldsymbol{C}}}} \\ &= \overline{\widehat{\dot{\boldsymbol{C}}}} + \overline{\widehat{\ddot{\boldsymbol{C}}}} + \overline{\widecheck{\dot{\boldsymbol{C}}}} + \overline{\widecheck{\ddot{\boldsymbol{C}}}} \\ &= \overline{\widehat{\boldsymbol{C}}} + \overline{\widecheck{\boldsymbol{C}}}\end{aligned}$$

As previously stated, the policy-level unallocated variables are defined to be zero.

$$\check{\dot{\boldsymbol{C}}}_j = 0 \qquad \text{implying} \qquad \dot{\boldsymbol{C}}_j = \widehat{\dot{\boldsymbol{C}}}_j$$
$$\check{\ddot{\boldsymbol{C}}}_j = 0 \qquad\qquad\qquad\qquad \ddot{\boldsymbol{C}}_j = \widehat{\ddot{\boldsymbol{C}}}_j$$

In addition to these relationships, two more are worthy of discussion.

1. Connecting Policy and Book Quantities
A book-level allocated quantity can be written as the sum of policy-level quantities.

$$\widehat{\boldsymbol{C}}_B = \sum_{j \in B} \boldsymbol{C}_j \qquad \widehat{\dot{\boldsymbol{C}}}_B = \sum_{j \in B} \dot{\boldsymbol{C}}_j \qquad \widehat{\ddot{\boldsymbol{C}}}_B = \sum_{j \in B} \ddot{\boldsymbol{C}}_j$$

However, unallocated book quantities cannot be written as the sum of policy quantities and stand on their own.

2. A Claims Specific Relationship
All the above relationships hold for any basic symbol, with the exception of premium which will be discussed. However, for claims it does not make sense to discuss unallocated pure costs for a book. To account for this, we define this term to be zero.

$$\check{\dot{\boldsymbol{C}}}_B = 0$$

This implies the following simplified relationships.

$$\dot{\boldsymbol{C}}_B = \widehat{\dot{\boldsymbol{C}}}_B \qquad \check{\boldsymbol{C}}_B = \check{\ddot{\boldsymbol{C}}}_B \qquad \boldsymbol{C}_B = \widehat{\dot{\boldsymbol{C}}}_B + \widehat{\ddot{\boldsymbol{C}}}_B + \check{\ddot{\boldsymbol{C}}}_B$$

One example of an unallocated pure cost for a book which is non-zero is recoveries from a stop-loss reinsurance treaty.

8.6 Estimator

The use of the hat accent to represent allocated costs conflicts with the standard convention from statistics for estimators. However, using the hat to represent allocated costs is more useful for three reasons.

1. It is intuitively related to the chosen symbol for a policy, Λ
2. Its "inverse" accent, the check, also has meaning
3. It can be combined with the average and pure/expense delimiters

So, a new notation is needed for estimators and estimates.

Definition The Estimator Delimiter

Description	Specifies an estimator of a quantity
Position	Below
Convention	A tilde signifies an estimator
	No tilde signifies the true quantity
Mnemonic	The tilde symbol evokes the meaning "roughly"

Example

$\underset{\sim}{\boldsymbol{C}}$ Estimator

Relationships
The relationship between the estimator for a distribution and an estimated value[16] is

$$\underset{\sim}{C} = \mathrm{E}\left[\underset{\sim}{\boldsymbol{C}}\right].$$

[16] This notation, combined with the proposed random variable notation, simplifies the distinction between true quantities, estimators, and estimates as discussed on Cross Validated.

8.7 Period Basis

Definition The Period Basis Delimiter
- Description: Constraint based on a type of period
- Position: Left superscript
- Convention: An upper case letter representing the type of basis followed by a letter representing the specific time, usually a year
- Mnemonic: The letters are chosen to resemble the period names

Example

${}^{CY}\boldsymbol{C}$	Total claim costs in calendar year Y
${}^{OY}\boldsymbol{C}$	Total claim costs in occurrence year Y
${}^{\Lambda Y}\boldsymbol{C}$	Total claim costs in policy year Y
${}^{RY}\boldsymbol{C}$	Total claim costs in report year Y
${}^{IT}\boldsymbol{C}$	Total claim costs for policies inforce at time T

Actuarial methods often require aggregating or averaging claims over many years. To accommodate such calculations, Y may be used as a parameter. For example, the report year claims summed from 2017 to 2019 is $\sum_{Y=2017}^{2019} {}^{RY}C$

Note: Formal definitions of these symbols also require definitions of the claim occurrence, report, and payment times, represented by ${}_{O}T_{j}$, ${}_{R}T_{j}$, ${}_{P}T_{j}$ and a policy's effective and expiration time, represented by ${}_{0}T_{j}$ and ${}_{1}T_{j}$. The short form is given to the paid time, $T_{j} = {}_{P}T_{j}$; this is consistent with the object delimiter discussion.

Short-Form

Not all period bases apply to all quantities, for example unallocated quantities cannot be grouped based on policy year. Calendar year is the only basis that applies all objects under all circumstances, so it is given a short form by omitting it.

$${}^{CY}\boldsymbol{C} = {}^{Y}\boldsymbol{C}$$

Additionally, if a problem involves a single year and it is unambiguously specified, we may omit the Y.

$${}^{Y}\boldsymbol{C} = \boldsymbol{C}$$

8.8 Type

Definition The Type Delimiter
- Description: Type of the basic symbol
- Position: Left subscript
- Convention: An upper case represents a type
  A primed upper case letter represents the opposite of the unprimed type
- Mnemonic: The letters are chosen to resemble the type names

The possible types vary with respect to basic symbols. The following are the most common types associated with claims.

Examples

${}_{P}C$	Total paid claim costs	${}_{P'}\boldsymbol{C}$	Total unpaid claim costs
${}_{R}\boldsymbol{C}$	Total reported claim costs	${}_{R'}\boldsymbol{C}$	Total unreported claim costs
${}_{O}\boldsymbol{C}$	Total incurred claim costs	${}_{O'}\boldsymbol{C}$	Total unincurred claim costs
${}_{U}\boldsymbol{C}$	Total ultimate claim costs		

Note: The symbol O is used for incurred claim costs for two reasons.
1. Another way of saying incurred claim costs is "costs for claims that have occurred."
2. The relation between incurred claims and the occurrence year period is emphasized by having a consistent symbol.

Obviously, all unincurred claims are unreported and unpaid. However, it is not useful to have such overlap in these symbols, so we introduce simplified definitions for claim types. Unreported claims are defined to be incurred and unreported claims. Similarly, unpaid claims are reported and unpaid claims.

Relationships
The equations relating the various types of claim costs are

$$
\begin{aligned}
{}_U\boldsymbol{C} &= {}_O\boldsymbol{C} + {}_{O'}\boldsymbol{C} \\
{}_O\boldsymbol{C} &= {}_R\boldsymbol{C} + {}_{R'}\boldsymbol{C} \\
{}_R\boldsymbol{C} &= {}_P C + {}_{P'}\boldsymbol{C}
\end{aligned}
$$

Short-Form
The ultimate value is the most important type because it encompasses all other types and applies to all basic symbols, so it is given a short form by omitting it.

$$ {}_U\boldsymbol{C} = \boldsymbol{C} $$

8.9 Level

Definition	The Level Delimiter
Description	Re-evaluates objects under conditions at different times
Position	Left Adjacent
Convention	A number sign whose left subscript indicates the time at which conditions are re-evaluated
Mnemonic	The number sign resembles parallelogram figures

Example
${}_T\#\boldsymbol{C}$ Accounts for discrete and continuous changes evaluated at time T

The mechanics of on-levelling differ significantly depending on the specifics of the situation at hand. This notation is intentionally general and solely represents the operation of on-levelling and not the precise calculational details[17]. This is analogous to the integration symbol, $\int$, which represents summing infinitesimal quantities but whose details and results differ based on circumstances.

In the simple case where on-leveling only involves a constant trend, it becomes

$$ {}_T\#\boldsymbol{C} = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{:n}(1+\tau)^{(T-\boldsymbol{T}_{:n})}. $$

Short-Form
When calculations require on-leveling, all quantities are usually bought to the level specified by a single time, usually the current-level time. As such, the time T is unambiguously specified and can be omitted to become

$$ \#\boldsymbol{C} = \ {}_T\#\boldsymbol{C}. $$

It is likely that the full-form will rarely be used.

[17] This is partially inspired by Michelbacher who noted that "we should recognize a definite point beyond which it is impractical to reduce our problems to formulae; our notation should not be too descriptive. We should decide in advance that a certain amount of narrative description must accompany our scientific discussions." [10]

8.10 Summary

All delimiters can be seen together in the following figure.

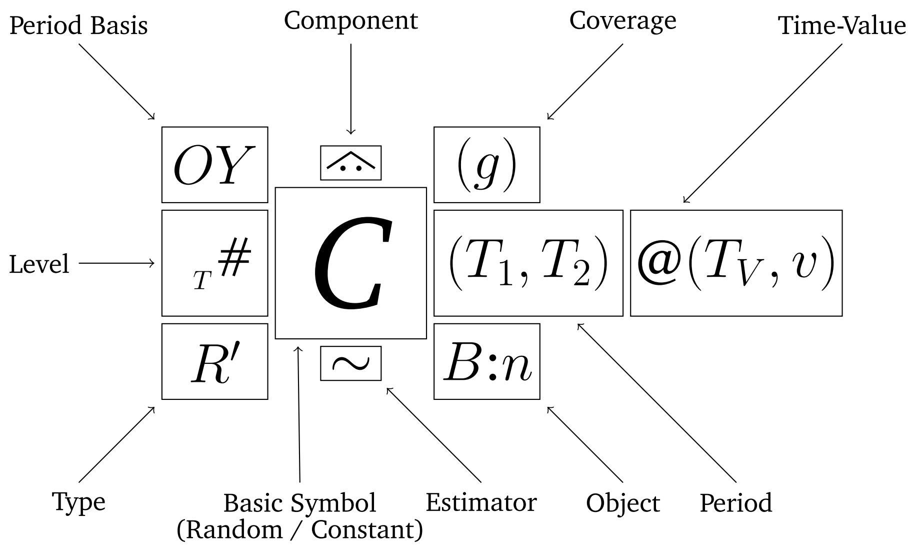

Fig. 4: A Schematic of the Proposed Notation

We now consider the unique properties of other basic terms.

8.11 Premiums

The delimiters that require further discussion for premiums are the type and component delimiters.

Type
The common types associated with premiums are as follows.

$_{W}P$	Written premium	$_{E}P$	Earned premium
$_{U}P$	Ultimate premium	$_{E'}P$	Unearned premium

Like claims, the ultimate type is given the short form $_{U}P = P$.

These same types also apply to units of exposures, X.

Other types of premium can also be defined depending on the need. For example, we may define $_{T}P$ and $_{C}P$ to represent theoretical and competitor premiums, respectively.

Component
The usual interpretation of pure/expense and allocated/unallocated does not apply to premiums. However, premiums must account for all other pure/expense and allocated/unallocated costs, so we use the component delimiter to represent these portions of the premium.

$\dot{P}$	The portion of the premium accounting for pure costs
$\ddot{P}$	The portion of the premium accounting for expense costs
$\widehat{P}$	The portion of the premium accounting for allocated costs
$\check{P}$	The portion of the premium accounting for unallocated costs

As with claims, the pure/expense and allocated/unallocated delimiters can be combined.

$\widehat{\dot{P}}$	The portion of the premium accounting for allocated pure costs
$\widecheck{\dot{P}}$	The portion of the premium accounting for unallocated pure costs
$\widehat{\ddot{P}}$	The portion of the premium accounting for allocated expense costs
$\widecheck{\ddot{P}}$	The portion of the premium accounting for unallocated expense costs

Together, these form the following relationships

$$\begin{aligned} P &= \dot{P} + \ddot{P} \\ &= \widehat{\dot{P}} + \widecheck{\dot{P}} + \widehat{\ddot{P}} + \widecheck{\ddot{P}} \\ &= \widehat{\dot{P}} + \widehat{\ddot{P}} + \widecheck{\dot{P}} + \widecheck{\ddot{P}} \\ &= \widehat{P} + \widecheck{P} \end{aligned}$$

8.12 Underwriting Expenses

The delimiters that require further discussion for underwriting expenses are the component delimiter and the coverage delimiter.

Component
Underwriting expenses are solely an expense quantity and there exists no pure counterpart. This leads to the following underwriting-specific simplifications.

$$\begin{aligned} \dot{U} &= 0 & & & U &= \ddot{U} \\ \widehat{\dot{U}} &= 0 & &\text{implying} & \widehat{U} &= \widehat{\ddot{U}} \\ \widecheck{\dot{U}} &= 0 & & & \widecheck{U} &= \widecheck{\ddot{U}} \end{aligned}$$

Coverage
There can be different categories of underwriting expenses, the most common of which are the following.

1. Commissions & Brokerage
2. Other Acquisition
3. General
4. Taxes, Licenses, & Fees

Differentiating kinds of underwriting expenses is done using the coverage delimiter.

$U^{(g:c)}$	Underwriting expenses for coverage g relating to expense category c.

The indices, g and c, can be replaced with slightly longer versions to aid in recognition. For example, consider a physical damage coverage of an auto insurance policy, the relevant quantities can be written as follows.

	Underwriting expenses for physical damage coverage relating to ...
$U^{(pd:cb)}$	... to commissions & brokerage.
$U^{(pd:oa)}$	... other acquisition costs.
$U^{(pd:g)}$	... to general costs.
$U^{(pd:tlf)}$	... taxes, licenses, and fees.

The individual practitioner may choose the indices he or she wishes to use that simplifies their problems. In either case, we have the following two relationships.

$$U^{(g)} = \sum_c U^{(g:c)} \quad \text{Total underwriting expenses for coverage } g \text{ across all expense categories}$$

$$U^{(:c)} = \sum_g U^{(g:c)} \quad \text{Total underwriting expenses for expense category } c \text{ across all coverages}$$

8.13 Recoveries

Like underwriting expenses, there can be multiple categories of recoveries, such as reinsurance, salvage, and subrogation; they differentiated in the same way by using the component delimiter.

From the perspective of an insurer, recoveries are cash inflows, whereas their expenses are outflows, so the relationship between total, pure, and expenses costs are as follows.

$$\boldsymbol{R} = \dot{\boldsymbol{R}} - \ddot{\boldsymbol{R}}$$

In the context of reinsurance, the pure costs, $\dot{\boldsymbol{R}}$, represent claim payments from the reinsurer to the insurer, and the expense costs, $\ddot{\boldsymbol{R}}$, represent premium payments from the insurer to the reinsurer. This notation, combined with the notation for layers, allows us to express various forms of reinsurance concisely.

Proportional Reinsurance
When a reinsurer assumes a portion of each individual claim, represented by the apportionment ratio a, the pure costs are expressed as

$$\dot{\boldsymbol{R}}_j = a\boldsymbol{C}_j.$$

This is an allocated cost and the total book cost is the sum of the individual costs, so we also have the following relationships.

$$\hat{\dot{\boldsymbol{R}}}_j = \dot{\boldsymbol{R}}_j \qquad \check{\dot{\boldsymbol{R}}}_j = 0 \tag{10}$$

$$\hat{\dot{\boldsymbol{R}}}_B = \dot{\boldsymbol{R}}_B \qquad \check{\dot{\boldsymbol{R}}}_B = 0 \qquad \text{where} \quad \dot{\boldsymbol{R}}_B = \sum_{j \in B} \dot{\boldsymbol{R}}_j \tag{11}$$

Excess Reinsurance: Per Risk
When a reinsurer assumes a portion of the excess layer for each claim, the pure costs are expressed as

$$\dot{\boldsymbol{R}}_j = a\langle l_0 | \boldsymbol{C}_j | l_1 \rangle.$$

This is also an allocated cost and the above relations, (10) and (11), hold as well.

Excess Reinsurance: Aggregate
When a reinsurer assumes a portion of the excess layer of the aggregate claims, the pure costs are

$$\dot{\boldsymbol{R}}_B = a\langle l_0 | \boldsymbol{C}_B | l_1 \rangle.$$

This is an unallocated cost because when the lower limit is breached there is no definitive method to determine which policies account for the excesses. This is an example of an unallocated pure costs and results in the following expressions.

$$\hat{\dot{\boldsymbol{R}}}_j = 0 \qquad \check{\dot{\boldsymbol{R}}}_j = 0 \qquad \dot{\boldsymbol{R}}_j = 0 \tag{12}$$

$$\hat{\dot{\boldsymbol{R}}}_B = 0 \qquad \check{\dot{\boldsymbol{R}}}_B = \dot{\boldsymbol{R}}_B \tag{13}$$

Excess Reinsurance: Per Event
When a reinsurer assumes a portion of the excess layer of aggregate claims for an individual event, the pure costs are

$$\dot{\boldsymbol{R}}_{B:e} = a\langle l_0 | \boldsymbol{C}_{B:e} | l_1 \rangle.$$

In the above, we made use of an extension of the object delimiter in which $\boldsymbol{C}_{B:e}$ represents the total claims for event e, each of which is comprised of individual claims $\boldsymbol{C}_{B:e:n}$ having the following relationship.

$$\boldsymbol{C}_{B:e} = \sum_{n=1}^{\boldsymbol{N}} \boldsymbol{C}_{B:e:n}$$

Like aggregate excess reinsurance, these costs are unallocated the relationships, (12) and (13), hold.

Layer Reinsurance
When a reinsurer assumes a different portion of different layers, the pure costs are

$$\begin{aligned}\dot{\boldsymbol{R}} &= a_1\langle l_0|\boldsymbol{C}|l_1\rangle + a_2\langle l_1|\boldsymbol{C}|l_2\rangle + \cdots + a_L\langle l_{L-1}|\boldsymbol{C}|l_L\rangle \\ &= \sum_{k=1}^{L} a_k\langle l_{k-1}|\boldsymbol{C}|l_k\rangle.\end{aligned}$$

8.14 Investment Income

Like recoveries, investment income is a cash inflow, whereas their expenses are outflows, so the relationship between total, pure, and expenses costs are as follows.

$$I = \dot{I} - \ddot{I}$$

Property and casualty insurance policies typically do not have an investment component, so the investment income is an unallocated amount, $\check{I}$. Many forms of life insurance policies have an investment component, and in these cases, the investment income can be considered an allocated amount, $\widehat{I}$.

8.15 Reserves

The type delimiter is used to specify the most common reserve types.

${}_{O'}V$	Unearned Premium Reserve
${}_{R'}V$	Pure IBNR Reserve
${}_{P'}V$	IBNER Reserve

Note: A reserve is never a random variable, so the symbol V should never be bolded.

8.16 Claim Count

The following delimiters do not apply to claim counts.

1. Time-Value
2. Pure/Expense Costs
3. Allocated/Unallocated Costs

Type
The claim count types are the types associated with claims and two additional types: closed and open.

${}_{C}N$	Closed claim count
${}_{C'}N$	Open claim count

8.17 The Fundamental Insurance Equation

The fundamental insurance equation is true for any component delimiter and its object form is as follows.

$$P_B = C_B + U_B - R_B - I_B + \Pi_B$$

$$\dot{P}_B = \dot{C}_B + \dot{U}_B - \dot{R}_B - \dot{I}_B + \dot{\Pi}_B$$

$$\widehat{P}_B = \widehat{C}_B + \widehat{U}_B - \widehat{R}_B - \widehat{I}_B + \widehat{\Pi}_B$$

To succinctly represent all possible versions of this equation, we introduce the ring component symbol. A ring indicates that an equation holds for all component delimiters.

$$\mathring{P}_B = \mathring{C}_B + \mathring{U}_B - \mathring{R}_B - \mathring{I}_B + \mathring{\Pi}_B$$

9 Secondary Basic Terms

There are two sets of secondary basic terms.

1. Same-Object Ratios
2. Different-Object Ratios

9.1 Same-Object Ratios

Same-object ratios are ratios of different primary basic terms related to the same object. For example, claims-ratio is a same-object ratio because it is a ratio of claims to premium for the same object, a book or policy.

The following are the most common same-object ratios and the symbols chosen to represent them.

Frequency	F	Premium Ratio	ρ	Development Factor	D
Severity	S	Expense Ratio	e		

All previously discussed delimiting terms apply to same-object ratios; however, we discuss some extensions of their usage to deal with ratios specifically.

9.1.1 Frequency & Severity

Frequency and severity are defined as follows.

$$F = \frac{N}{X} \qquad \text{Average Claim Count per Unit Exposure}$$

$$S = \frac{C}{N} \qquad \text{Average Claim Cost per Claim Count}$$

Note: $\overline{N}$ also represents frequency. This is the only instance in which two completely different symbols represent the same concept. Frequency is an important concept and if the symbol F were used for another concept, it may cause confusion, so an exception is made in this case.

These definitions imply the following relationship.

$$FS = \overline{C}$$

The Period Basis Delimiter
When taking the ratio of two basic terms, they may or may not have the same period basis. When the two terms have the same basis, no modifications to the notation are required. To account for situations in which the two terms have different period bases we use a backslash.

$$^{CY}F = \frac{^{CY}N}{^{CY}X} \qquad ^{O/\Lambda Y}F = \frac{^{OY}N}{^{\Lambda Y}X}$$

The Type Delimiter
The same idea applies to the type delimiter as shown in the following two examples.

$$_{U}F = \frac{_{U}N}{_{U}X} \qquad _{R/E}F = \frac{_{R}N}{_{E}X}$$

9.1.2 Premium Ratios

A premium ratio is the ratio of a basic term relative to the total premium. The basic term is specified using the object delimiter. For example, the notation for the ratio of claims to premium for a book and a policy are as follows.

$$\rho_{B|C} = \frac{C_B}{P_B} \qquad \rho_{j|C} = \frac{C_j}{P_j}$$

The symbol | is used rather than : to separate the object and the basic symbol because the symbol : is reserved for uses that imply summations.

Since premium ratios are more commonly used for books, we define the short-form version in which the book object may be omitted.

$$\rho_C = \rho_{B|C}$$

Below are more examples of premium ratios.

$$\rho_R = \frac{R_B}{P_B} \qquad \rho_U = \frac{U_B}{P_B} \qquad \rho_I = \frac{I_B}{P_B} \qquad \rho_\Pi = \frac{\Pi_B}{P_B}$$

When a premium ratio involves different period bases or types, the same convention discussed in the previous section is used.

The Component Delimiter
The component delimiter applies only to the numerator as shown in the examples below.

$$\dot{\rho}_C = \frac{\dot{C}_B}{P_B} \qquad \widehat{\dot{\rho}}_R = \frac{\widehat{\dot{R}}_B}{P_B} \qquad \check{\rho}_U = \frac{\check{U}_B}{P_B} \qquad \ddot{\rho}_I = \frac{\ddot{I}_B}{P_B}$$

The fundamental insurance equation can be written in terms of premium ratios as follows.

$$1 = \rho_C + \rho_U - \rho_R - \rho_I + \rho_\Pi$$

9.1.3 Expense Ratio

An expense ratio is the ratio of a basic term's expense cost relative to its total or pure cost. When the pure component is used in the accent, it applies only to the denominator. In contrast, when the allocated/unallocated component is used, it applies to both the numerator and denominator.

$$e_C = \frac{\ddot{C}_B}{C_B} \qquad \dot{e}_R = \frac{\ddot{R}_B}{\dot{R}_B} \qquad \widehat{e}_C = \frac{\widehat{\ddot{C}}_B}{\widehat{C}_B} \qquad \check{\dot{e}}_R = \frac{\check{\ddot{R}}_B}{\check{\dot{R}}_B}$$

9.1.4 Development Factor

A development factor is a ratio of the same basic term with different cumulative periods. The most common development factor is associated with claims and applicable to loss reserving.

A development factor can be expressed using absolute time or relative time.

$$D_C(T_1, T_2) = \frac{C_B(T_0, T_2)}{C_B(T_0, T_1)} \qquad D_C(t_1, t_2) = \frac{C_B(t_2)}{C_B(t_1)}$$

Relative times are more commonly used in connection to development factors, for example in the chain ladder methods, so the following discussion uses relative times. A development factor can be considered a random variable by making the variable bold, $\boldsymbol{D}$.

Because development factors are most commonly associated with claims, it may be omitted as a short-form.

$$D(t_1, t_2) = D_C(t_1, t_2)$$

When a development factor is specified using two times, it is an age-to-age factor. The notation for an age-to-ultimate factor uses a single time and is defined as follows.

$$D(t_1) = \frac{C_B}{C_B(t_1)}$$

The type and period basis delimiters aid in distinguishing between various development factors.

${}^{CY}_{R}D(t_1, t_2)$	Calendar year Y reported age-to-age factor from t_1 to t_2
${}^{OY}_{P}D(t_1, t_2)$	Occurrence year Y paid age-to-age factor from t_1 to t_2
${}^{CY}_{R}D(t_1)$	Calendar year Y reported age-to-ultimate factor for time t_1

Selections are specified using an S in the period basis delimiter and can be written in terms of historical development factors. For example, suppose the selected occurrence year age-to-age factor for paid claims from 12 months to 24 months was the average of the historical development factors over the prior three years; this can be written as follows.

$$ {}^{OS}_{P}D(12,24) = \frac{1}{3}\sum_{Y=2017}^{2019} {}^{OY}_{P}D(12,24) $$

9.2 Different-Object Ratios

Different-object ratios are ratios of the same primary basic term of two different objects. For example, a claim relativity is a different-object ratio because it is the ratio of the expected claims of two different policies. Different-object ratios are generally referred to as relativities.

Before discussing the relativities we must first introduce the characteristic vector to differentiate basic terms of different objects, we use claims as an example.

The characteristics of a claims distribution depend on the sub-field of insurance and the specific product; in life insurance this may include age and smoker-status, whereas for an auto insurance policy this may include vehicle-type and odometer reading. The claims characteristics are represented by the vector ${}_{C}\vec{k}$ and the claims variable is written in terms as

$$ \boldsymbol{C}_j = \boldsymbol{C}({}_{C}\vec{k}_j). $$

As a short-form, we may omit the type C in the above expression since it is enclosed in C, making it obvious that it refers to claims.

$$ \boldsymbol{C}_j = \boldsymbol{C}(\vec{k}_j) $$

The same idea applies to other sets of characteristics.

${}_{X}\vec{k}_j$	Exposure characteristics of policy j, for example car-age
${}_{\Lambda}\vec{k}_j$	Policy characteristics of policy j, for example deductible
${}_{P}\vec{k}_j$	Premium characteristics of policy j, for example car-age and deductible
${}_{C}\vec{k}_j$	Claim characteristics of policy j, for example car-age, deductible, and type of accident

There is overlap between these sets of characteristics as alluded to in the above examples; premium characteristics are comprised of exposure and policy characteristics, and claim characteristics encompass all those mentioned.

We now continue the discussion of relativities using claims as an example. In particular, we discuss two kinds of relativities: the expected and variance relativities.

To facilitate comparisons between different objects, a base class must be chosen as a reference. The characteristics of the base class are denoted by $\vec{b}$ and its expected value and variance are defined as

$$ \beta_E = \mathrm{E}\left[\boldsymbol{C}(\vec{b})\right] \quad \text{and} \quad \beta_V = \mathrm{Var}\left[\boldsymbol{C}(\vec{b})\right]. $$

The expected and variance claims-relativities of a policy j having characteristics $\vec{k}_j$ are defined as

$$ \epsilon(\vec{k}_j) = \frac{\mathrm{E}\left[\boldsymbol{C}(\vec{k}_j)\right]}{\mathrm{E}\left[\boldsymbol{C}(\vec{b})\right]} \quad \text{and} \quad \nu(\vec{k}_j) = \frac{\mathrm{Var}\left[\boldsymbol{C}(\vec{k}_j)\right]}{\mathrm{Var}\left[\boldsymbol{C}(\vec{b})\right]}. $$

If there are no interactions between individual characteristics, the relativities are separable and can be written as

$$\epsilon(\vec{k}_j) = \epsilon(k_j^{(1)})\epsilon(k_j^{(2)}) \cdots \epsilon(k_j^{(K)}) \qquad \text{and} \qquad \nu(\vec{k}_j) = \nu(k_j^{(1)})\nu(k_j^{(2)}) \cdots \nu(k_j^{(K)}).$$

The expected and variance claims-relativities can be rewritten in terms of the base class as

$$\mathrm{E}\left[\boldsymbol{C}(\vec{k}_j)\right] = \beta_E \cdot \epsilon(\vec{k}_j) \qquad \text{and} \qquad \mathrm{Var}\left[\boldsymbol{C}(\vec{k}_j)\right] = \beta_V \cdot \nu(\vec{k}_j).$$

10 Conclusion

This proposal is simply that: a proposal; despite careful deliberations, it certainly requires input, modifications, and discussion from the actuarial community to account for any overlooked aspects.

The adoption of a standard notation must be a community consensus; actuaries at the Second International Congress understood this fact. Before the final vote on standardizing life notation, George King said that "he earnestly hoped there would be unanimity. In fact, were there not unanimity, there could not be a Universal Notation; because those who disagreed would go on using their own individual notation and there was no penalty which could be imposed to prevent them from doing so" [3].

The necessity of a standard notation is eloquently summarized by Maurice Ogborn. In 1966, the Institute of Actuaries awarded the Silver Medal to Ogborn and specifically cited his service as a member of the Committee on Notation and his 1936 prize essay on actuarial notation [51]; it begins as follows [52].

> "The notation used by writers on any scientific subject seldom gets the consideration from them that it deserves. In consequence a mass of symbols gradually accumulates having little relation to each other or to the symbols in analogous subjects. An attempt to bring about order in this chaos is looked on as pedantic, people fearing that a change in notation may create more confusion than it disentangles.
>
> "I think this point of view is rather short-sighted. A good notation will enable ideas to be set out more concisely without loss of accuracy or lucidity. Indeed, if the symbols are well chosen they may, by their very form, suggest other relationships which would not otherwise have been discovered.
>
> "On the other hand, a poor notation will only make confusion worse confounded; so that writers would make their meaning as clear by setting out their work in full in words as by using the symbols of the notation. Anyone who doubts the benefits we have received in actuarial work from the system of notation, may compare it with the system given by Joshua Milne in his book on *Annuities and Assurances*, 1815, Introduction [53], pp. liv and lv."
> — Maurice Ogborn (1936)

More recently, when Chris Daykin became the President of the Institute of Actuaries in 1994, he said, in his Presidential Address, that "we need to reassert that the actuarial profession is a mathematical discipline. It is the rigour of mathematics and the immense potential of mathematical modelling which give flavour to the role of the actuary" [54].

To reassert the mathematical nature of actuarial science, improve communication at all levels, and promote innovation, we need a mathematical language that can be used by any person, from any nation, doing any actuarial work: a Universal Actuarial Notation.

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