An Application of Game Theory: Property Catastrophe Risk Load

by Donald F. Mango, FCAS

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Abstract

Two well-known methods for calculating risk load -- Marginal Surplus and Marginal Variance -- are applied to output from catastrophe modeling software. Risk loads for these "marginal methods" are calculated for sample new and renewal accounts. Differences between new and renewal pricing are examined. For new situations, both current methods allocate the full marginal impact of addition of a new account to that new account. For renewal situations, a new concept is introduced -- "renewal additivity". Neither marginal method is renewal additive. A new method is introduced, inspired by game theory, which splits the mutual covariance between any two accounts evenly between those accounts. The new method is extended and generalized to a proportional sharing of mutual covariance between any two accounts. Both new approaches are tested in new and renewal situations.

(1) Introduction

The calculation of risk load continues to be a topic of interest in the actuarial community -- see Bault [1] for a recent survey of well-known alternatives. One area where the CAS literature is somewhat scarce, and the need is great, is calculation of risk loads for property catastrophe insurance.

The new catastrophe modeling products produce modeled "occurrence size-of-loss distributions" for a series of simulated events. Using the occurrence size-of-loss distribution, one can easily calculate expected losses, loss variance and standard deviation. Two of the more well-known risk load methods from the CAS literature -- what I call "Marginal Surplus" (MS) from Kreps [3] and "Marginal Variance" (MV) from Meyers [6] -- use the marginal change in portfolio standard deviation (respectively variance) due to addition of a new account as a means to calculate the risk load for that new account. However, as we shall see, problems arise when we use these marginal methods in calculating the risk loads for the renewal of the accounts in a portfolio.

We apply the MV and MS methods to a simplified occurrence size-of-loss distribution, calculate risk loads both in assembling or building up a portfolio of risks, and in subsequently renewing that portfolio. Then we discuss the differences between build-up and renewal results.

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[1] *I would like to thank Eric Lemieux and Sean Ringsted for their support, editorial suggestions and review of early drafts. I would also like to thank Paul Kneuer for his thoughtful and insightful review which improved the paper.*

We then introduce a new concept to the theory of property catastrophe risk loads -- renewal additivity. However, the concept is not new to the field of game theory, where we will draw inspiration for a new approach.

We begin with a brief outline of the mechanics of catastrophe occurrence size-of-loss distributions, and the calculation of risk loads using the two marginal methods.

(2) The Catastrophe Occurrence Size-of-loss Distribution

For demonstration purposes throughout the paper, we will use a simplified version of an occurrence size-of-loss distribution. It captures the essence of typical catastrophe modeling software output, while keeping the examples understandable[2].

A series of modeled events denoted by identifier i are considered independent Poisson processes each with occurrence rate λ_i. To simplify the mathematics, following Meyers [6], we will employ the binomial approximation with probability of occurrence p_i [where $\lambda_i = -\ln(1 - p_i)$]. This is a satisfactory approximation for small λ_i [3].

For an individual account or portfolio of accounts, the model produces an expected loss for each event L_i. We will refer to a table containing the event identifiers i, the event probabilities p_i and modeled expected losses L_i as an "occurrence size-of-loss distribution."

From Meyers [6], the formulas for expected loss and variance are [Σ_i = sum over all events]:

$$E[L] = \Sigma_i \{ L_i * p_i \} \qquad [2.1]$$

$$Var[L] = \Sigma_i \{ L_i^2 * p_i * (1 - p_i) \} . \qquad [2.2]$$

The formula for covariance of an existing portfolio L (with losses L_i) and a new account n (with losses n_i) is :

$$Cov[L, n] = \Sigma_i \{ L_i * n_i * p_i * (1 - p_i) \} \qquad [2.3]$$

The total variance of the combined portfolio [L + n] is then

[2] In particular, we will only be considering single event or occurrence size-of-loss distributions. Many models also produce multi-event or aggregate loss distributions. Occurrence size-of-loss distributions only reflect the *largest* event which occurs in a given year. Aggregate loss distributions reflect the sum of losses for all events in a given year. Clearly, the aggregate table provides a more complete picture, but for purposes of our exposition here, the occurrence table works well and the formulas are substantially less complex.

[3] An event with a probability of 0.001 (typical of the more severe modeled events) would have λ = 0.0010005.

$$Var[L] + Var[n] + 2 * Cov[L, n] \quad [2.4]$$

(3) The Marginal Surplus (MS) Method

This is a translation to property catastrophe of the method described in Rodney Kreps' "Reinsurer Risk Loads from Marginal Surplus Requirements" [3].

Consider:

L_0 = losses from a portfolio before a new account is added
L_1 = losses from a portfolio after a new account is added
S_0 = Standard deviation of L_0
S_1 = Standard deviation of L_1

Borrowing from Mr. Kreps, assume needed surplus V is given by

z * *Standard Deviation of loss*[4] *- expected Return* [3.1]

where *z* is, to cite Mr. Kreps (p. 197), "a distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated." Then

$$V_0 = z * S_0 - R_0$$
$$V_1 = z * S_1 - R_1 \quad [3.2]$$

The difference in returns $R_1 - R_0 = r$, the risk load charged to the new account. The marginal surplus requirement is then

$$V_1 - V_0 = z * [S_1 - S_0] + r \quad [3.3]$$

We determine the risk load based on required return *y* on that marginal surplus, which is based on management goals, market forces and risk appetite. The MS risk load would be:

$$r = y * z / (1 + y) * [S_1 - S_0] \quad [3.4]$$

(4) The Marginal Variance (MV) Method

This is based on Glenn Meyers' 1995 CAS Discussion Paper program article "Managing the Catastrophe Risk" [6].

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4 Mr. Kreps sets needed surplus equal to z * standard deviation of return - expected return. If we assume premiums and expenses are invariant, then Var[Return] = Var[P - E - L] = Var[L].

For an existing portfolio L and a new account n, the MV risk load would be:

$$r = \lambda * \text{Marginal Variance of adding n to L}$$
$$= \lambda * \{ Var[n] + 2 * Cov[L, n] \} \quad [4.1]$$

where λ is a multiplier similar to y * z / (1 + y) from the MS method, although dimensioned to apply to variance rather than standard deviation[5].

(5) Building Up a Portfolio of 2 Accounts

Now we are prepared to apply the methods to the sample portfolio. Table A shows the occurrence size-of-loss distribution and risk load calculations for building up (assembling) a portfolio of 2 accounts, (X) and (Y). We assume (X) is written first, and is the only risk in the portfolio until (Y) is written.

(5.1) MS Method

Here is a summary of pertinent values from Table A for the Marginal Surplus method:

Table 5.1

Building Up (X) & (Y): Marginal Surplus	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Standard Deviation	4,429	356	4,785	4,785
(2) Risk Load Multiplier	0.33	0.33	-	0.33
(3) Risk Load = (1) * (2)	$1,461.71	$117.43	$1,579.14	$1,579.14

- Item (1) is the change in portfolio standard deviation from adding each account, or *marginal* standard deviation.

- Item (2) is the Risk Load multiplier of 0.33. Using Mr. Kreps' formula, a return on marginal surplus y of 20% and a standard normal multiplier z of 2.0 (2 standard deviations, corresponding to a cumulative non-exceedance probability of 97.725%) would produce a risk load multiplier of

$$y * z / (1 + y) = \quad 0.20 * 2 / 1.20 \quad = \quad 0.33 \text{ (rounded)} \quad [5.1]$$

- Item (3) is the Risk Load, the product of Items (1) and (2).

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5 Mr. Meyers develops a variance based risk load multiplier by converting a standard deviation based multiplier using the following formula:

λ = (Rate of Return * Std Dev Mult^2) / (2 * Avg Capital of Competitors)

Since (X) is the first account, the marginal standard deviation from adding (X) equals the standard deviation of (X) (Std Dev [X]) of 4,429. This gives a risk load of $1,461.71.

The marginal standard deviation from writing (Y) equals Std Dev [X + Y] - Std Dev [X], or $356, implying a risk load of $117.43.

The sum of these two risk loads (X) + (Y) is $1,461.71 + $117.43 = $1,579.14. This equals the risk load which this method would calculate for the combined account (X + Y).

(5.2) MV Method
Here is a summary of pertinent values from Table A for the Marginal Variance method:

Table 5.2

Building Up (X) & (Y): ***Marginal Variance***	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Variance	19,619,900	3,279,059	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,353.02	$226.13	$1,579.14	$1,579.14

• Item (1) is the change in portfolio variance from adding each account, or *marginal* variance.

• Item (2) is the Variance Risk Load multiplier λ of 0.000069. To simplify comparisons between the two methods (recognizing the difficulty of selecting a MV-based multiplier[6]), I converted the MS multiplier to a MV basis by dividing by Std Dev [X + Y]:

$$\lambda = 0.33 / 1{,}579.14 = 0.000069 \quad [5.2]$$

This means the total risk load calculated for the portfolio by the two methods will be the same, although the individual risk loads for (X) and (Y) will differ between the methods.

• Item (3) is the Risk Load, the product of Items (1) and (2).

Since (X) is the first account, the marginal variance from adding (X) equals the variance of (X) (Var [X]) of 19,619,900. This gives a risk load of $1,353.02.

The marginal variance from writing (Y) equals Var [X + Y] - Var [X], or $3,279,059, implying a risk load of $226.13.

[6] Mr. Meyers [6] (p.124) admits that in practice "it might be difficult for an insurer to obtain the (lambdas) of each of its competitors." He goes on to suggest an approximate method to arrive at a usable lambda based on required capital being "Z standard deviations of the total loss distribution."

The sum of these two risk loads (X) + (Y) is $1,353.02 + $226.13 = $1,579.14. This equals the risk load which this method would calculate for the combined account (X + Y).

(6) Renewing the Portfolio of 2 Accounts

Table B shows the natural extension of the Build-up scenario -- renewal of these 2 accounts, in what could be termed a "static" or "steady state" portfolio (one with no new entrants).

As for applying these methods in the renewal scenario, renewing policy (X) is assumed equivalent to adding (X) to a portfolio of (Y); renewing (Y) is assumed equivalent to adding (Y) to a portfolio of (X).

(6.1) MS Method
Here is a summary of pertinent values from Table B for the Marginal Surplus method:

Table 6.1

Renewing (X) & (Y): Marginal Surplus	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Standard Deviation	4,171	356	4,526	4,785
(2) Risk Load Multiplier	0.33	0.33	-	0.33
(3) Risk Load = (1) * (2)	$1,376.27	$117.43	$1,493.70	$1,579.14
(4) Build-up Risk Load	$1,461.71	$117.43	$1,579.14	$1,579.14
(5) Difference	($85.45)	$0	($85.45)	$0

The marginal standard deviation for adding (Y) to (X) is 356, same as it was during Build-up -- see Section (5.1). The risk load of $117.43 is also the same.

However, adding (X) to (Y) gives a marginal standard deviation of Std Dev [X + Y] - Std Dev [Y] , or 4,171. This gives a risk load for (X) of $1,376.27, which is (85.45) **less** than $1,461.71, the risk load for (X) calculated in Section (5.1).

The sum of these two risk loads is $1,376.27 + $117.43 = $1,493.70. This is also (85.45) **less** than the total risk load from Section (5.1).

(6.2) MV Method
Here is a summary of pertinent values from Table B for the Marginal Variance method:

Table 6.2

Renewing (X) & (Y): Marginal Variance	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Variance	22,521,000	3,279,059	25,800,059	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,553.08	$226.13	$1,779.21	$1,579.14
(4) Build-up Risk Load	$1,353.02	$226.13	$1,579.14	$1,579.14
(5) Difference	$200.06	$0	$200.06	$0

The marginal variance for adding (Y) to (X) is 3,279,059, same as it was during Build-up -- see Section (5.2). The risk load of $226.13 is also the same.

However, adding (X) to (Y) gives a marginal variance of Var [X + Y] - Var [Y], or 22,521,000. The risk load is now $1,553.08, which is $200.06 **more** than the $1,353.02 calculated in Section (5.2).

The sum of these two risk loads is $1,553.08 + $226.13 = $1,779.21. This is also $200.06 **more** than the total risk load from Section (5.2).

(7) Exploring the Differences Between New and Renewal

Why are the total Renewal risk loads different from the total Build-up risk loads?

(7.1) MS Method

In Section (5.1) Build-up, the marginal standard deviation for (X), ΔStd Dev [X], was :

$$\Delta\text{Std Dev }[X] = \text{Std Dev }[X] = \text{SQRT}\,[\,\Sigma_i\ \{X_i^2 * p_i * (1 - p_i)\}\,], \qquad [7.1]$$

(X_i = *modeled losses for X for event i*)

while in Section (6.1) Renewal, the marginal standard deviation was

$$\Delta\text{Std Dev }[X] = \text{Std Dev }[X + Y] - \text{Std Dev }[Y] = \text{SQRT}\,[\,\Sigma_i\ \{(X_i+Y_i)^2 * p_i * (1 - p_i)\}\,] - \text{SQRT}\,[\,\Sigma_i\ \{Y_i^2 * p_i * (1 - p_i)\}\,] \qquad [7.2]$$

For positive Y_i, this value is less than Std Dev [X][7]. Therefore, we *would* expect the Renewal risk load to be less than the Build-up.

[7] For example, assume Var [X] = 9, Var [Y] = 4, Cov [X, Y] = 1.5; then
ΔStd Dev [X] = Sqrt(Var [X]) = Sqrt (9) = 3 *for X alone*
ΔStd Dev [X] = Sqrt(9 + 4 + 2*1.5) - Sqrt(4) = 4 - 2 = 2 < 3. *for X added to Y*

Unfortunately, when the MS method is applied in the renewal of all the accounts in a portfolio, the sum of the individual risk loads will be less than the total portfolio standard deviation times the multiplier. This is because the sum of the marginal standard deviations (found by taking the difference in portfolio standard deviation with and without each account in the portfolio) is less than the total portfolio standard deviation[8]. This is because **the square root operator is "sub-additive"**: the square root of a sum is less than the sum of the square roots[9].

(7.2) <u>MV Method</u>
In Section (5.2) Build-up, the marginal variance $\Delta Var\,[X]$ was

$$\begin{aligned} \Delta Var\,[X] &= Var\,[X] \\ &= \Sigma_i\ \{ X_i^2 * p_i * (1 - p_i) \}, \end{aligned} \qquad [7.3]$$

while in Section (6.2) Renewal the marginal variance was

$$\begin{aligned} \Delta Var\,[X] &= Var\,[X + Y] - Var\,[Y] \\ &= \{ Var\,[X] + 2 * Cov\,[X, Y] + \cancel{Var\,[Y]} \} - \cancel{Var\,[Y]} \\ &= Var\,[X] + 2 * Cov\,[X, Y] \\ &> Var\,[X]. \end{aligned} \qquad [7.4]$$

Since $2 * Cov\,[X, Y]$ is greater than zero, we *would* expect the Renewal risk load to be greater than the Build-up.

However, when the MV method is applied in the renewal of all the accounts in a portfolio, the sum of the individual risk loads will be more than the total portfolio variance times the multiplier. This is because the sum of the marginal variances (found by taking the difference in portfolio variance with and without each account in the portfolio) is greater than the total portfolio variance. **The covariance between any two risks in the portfolio is double counted**: when each account renews, it is allocated the full amount of its shared covariance with all the other accounts.

(8) <u>A New Concept: Renewal Additivity</u>

The renewal scenarios point out that these two methods are not what I call "**<u>renewal additive</u>**," defined as follows:

> For a given portfolio of accounts, a risk load method is **<u>renewal additive</u>** if the sum of the renewal risk loads calculated for each component account equals the risk load calculated when the combined accounts are treated as a single account.

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8 The same issue is raised in Mr. Gogol's discussion [2].

9 For example, Sqrt[9 + 16] < Sqrt[9] + Sqrt[16].

Neither the MS nor the MV method is renewal additive: MS because the square root operator is sub-additive; MV because the covariance is double counted. In order for them to be renewal additive, one must assume an **entry order** for the accounts.

It's a puzzling predicament. We apply the risk load formula for the renewal of account (X). The formula makes sense for the renewal of account (X). It also makes sense for the renewal of account (Y). However, the portfolio total does not make sense. We could say that in the renewal context, these methods were "**individually rational**" yet the total was not "**collectively rational**".

I chose these terms deliberately as a segue to the next section. They come from the field of game theory. These concepts and others (including additivity) have been studied extensively by game theorists, and their results will provide us with inspiration for a new approach.

(9) A New Approach from Game Theory

I focused on ideas in two papers by Jean Lemaire: "An Application of Game Theory: Cost Allocation" [4], and "Cooperative Game Theory and Its Insurance Applications" [5]. In both papers, Mr. Lemaire considers the insurance applications of results from "cooperative games with transferable utilities"[10].

The material can be daunting. To facilitate the discussion, I will combine and paraphrase the formal game theory definitions from both of Mr. Lemaire's papers, then follow with translations to our problem[11].

Basics

> "A n-person cooperative game with transferable utilities is a pair [N, v(S)] where N = {1, 2, ..., n} is the set of the players, and v(S), the characteristic function of the game, is a super-additive[12] set function that associates a real number v(S) with each coalition S of players" ([4], p. 68).

[10] Citing Mr. Lemaire [5] (p.20) : "Cooperative game theory analyzes those situations where participants' objectives are partially cooperative and partially conflicting. It is in the participants' interest to cooperate, in order to achieve the greatest possible total benefits. When it comes to sharing the benefits of cooperation, however, individuals have conflicting goals.... Participants are negotiating about sharing a given commodity (such as money or political power) which is fully transferable between players and evaluated in the same way by everyone.... For this reason, the class of games defined here is called 'Cooperative games with transferable utilities.'"

In our case, the conflicting goals arise because all but the largest risks must have catastrophe coverage, and must go for this coverage to an insurance company. Insurance companies write many such risks, which means they have loss covariance created by the pooling of risks exposed to the same potential catastrophic events. The desire for coverage conflicts with the desire to be allocated the least covariance.

[11] Those wishing a more detailed explanation are strongly encouraged to read Mr. Lemaire's papers.

[12] Super-additivity is defined as follows: for S, T any two disjoint coalitions, and a characteristic function v, super-additivity implies v(S) + v(T) <= v(S union T).

Translation:
- Player = account.
- Coalition S = portfolio.
- Characteristic function v(S) = portfolio variance (super-additive because of the covariance component).

Imputation, Individual rationality, additivity

"An **imputation** is a vector $y = (y_1, ..., y_n)$ such that $y_i >= v(i)$ for every i, and $\Sigma_{i=1 \text{ to } n} y_i = v(N)$" ([5] p. 68).

Translation:
- Imputation = allocation of the coalition total value v(N) back to the individual members.

- The first condition ($y_i >= v(i)$ for every i) is known as "**individual rationality**" -- each member's allocation y_i is no smaller than its value would be were it on its own (= v(i)).

- The second condition ($\Sigma_{i=1 \text{ to } n} y_i = v(N)$) is known as "**additivity**" -- the sum of the individual allocations must add up to the coalition total value.

In our problem, the imputation is each account's marginal variance (under the MV method) from adding it to the remainder of the portfolio. This imputation is individually rational, since the allocations are larger than the individual account variances because of the covariance component. However, as we have seen, it is **not** additive -- the sum of the individual allocations (marginal variances) is greater than the total variance.

Collective rationality and the Core

"An imputation is **collectively rational** if there is no sub-coalition S' under which the players are better off than they were under S.

"The **core** of the game is the set of all collectively rational imputations." ([5], p. 25)

Translation:
- Collectively rational = the coalition is stable -- there is no incentive for players to split off and form factions.

- The core sets the boundaries for possible, stable allocations.

Shapley value

"The **Shapley value** is the center of gravity of the core's extremal points." ([4], p. 72)

Translation:
The Shapley value is the only allocation which satisfies the following three axioms ([4], p. 69):

1. Symmetry (Order-independence) - for all permutations P(S) of accounts in a portfolio S, c(S) = c(P(S)). Knowing the combination of accounts is sufficient to have an additive allocation.

2. Inessential Players (Uncorrelated accounts) - if an account generates no covariance with the existing portfolio, it is simply allocated its own variance, and nothing more.

3. Additivity - allocations from distinct games should be additive. This particular condition has no parallel in our situation.

Only one allocation method satisfies these three axioms -- the "**Shapley value**". It equals the average allocation taken over all possible **entrance permutations** -- the different orders in which a new member could have been added to the coalition[13] (i.e. a new account could have been added to a portfolio).

For example, if we had a portfolio of accounts (A), (B), and (C), and we want to add a new account (D), we could consider the marginal variance for adding (D) in all the following entrance permutations:

Table 9.1
Entry Permutations for Account D

(1)	(2)	(3)	(4)
Permutation #	**Entry Order**	**After...**	**Marginal Variance**
1	First	-	Var [D]
2	Second	After (A)	Var [D] + 2*Cov [D, A]
3	"	After (B)	Var [D] + 2*Cov [D, B]
4	"	After (C)	Var [D] + 2*Cov [D, C]
5	Third	After (AB)	Var [D] + 2*Cov [D, A] + 2*Cov [D, B]
6	"	After (AC)	Var [D] + 2*Cov [D, A] + 2*Cov [D, C]
7	"	After (BC)	Var [D] + 2*Cov [D, B] + 2*Cov [D, C]

13 Mr. Lemaire [5] provides this more complete definition of the Shapley value (p. 29): "The Shapley value can be *interpreted* as the mathematical expectation of the admission value, when all orders of formation of the grand coalition are equiprobable. In computing the value, one can assume, for convenience, that all players enter the grand coalition one by one, each of them receiving the entire benefits he brings to the coalition formed just before him. All orders of formation of N are considered and intervene with the same weight 1/n! in the computation. The combinatorial coefficient results from the fact that there are (s-1)!(n-s)! ways for a player to be the last to enter coalition S: the (s-1) other players of S and the (n-s) players of N\S (*those players in N which are not in S - DM*) can be permuted without affecting i's position."

8	Fourth	After (ABC)	Var [D] + 2*Cov [D, A] + 2*Cov [D, B] + 2*Cov [D, C]

The Shapley value is the straight average of Column (4) Marginal Variance over the eight permutations:

Shapley Value = { Sum [Column (4)] } / 8 [9.1]

= { 8*Var [D] +
8*Cov [D, A] +
8*Cov [D, B] +
8*Cov [D, C] } / 8

= Var [D] + Cov [D, A] + Cov [D, B] + Cov [D, C]

Or, to generalize, given

L = losses for existing portfolio
n = losses for new account

Shapley Value = Var [n] + Cov [L, n]. [9.2]

Before seeing this result, we might have been concerned about the practicality of this approach -- how much computational time might be required to calculate all the possible entrance permutations for a portfolio of thousands of accounts? This simple reduction formula eliminates those concerns. The Shapley value is as simple to calculate as the marginal variance.

Comparing the Shapley value to the marginal variance formula from Section 4:

Marginal Variance = Var [n] + **2** * Cov [L, n], [9.3]

we note the Shapley value only takes 1 times the covariance of the new account and the existing portfolio.

We can also calculate the Shapley value under the marginal standard deviation method. However, due to the complex nature of the mathematics -- differences of square roots of sums of products -- no simplifying reduction formula was immediately apparent[14].

Therefore, we will focus going forward on the MV method and the variance-based Shapley value. Life will be much easier (mathematically) working with the variances,

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14 Please contact the author if you can successfully reduce formulas involving the average of the difference of square roots of sums of products.

and we lose very little by choosing variance. Citing Mr. Bault ([1], p. 82), from a risk load perspective, "both [variance and standard deviation] are simply special cases of a unifying covariance framework." In fact, Mr. Bault goes on to suggest "in most cases, the 'correct' answer is a marginal risk approach that incorporates covariance"[15].

(10) Sharing the Covariance

The risk load question, framed in a game-theoretical light, has now become:

How do accounts share their mutual covariance for purposes of calculating risk load?

The Shapley method answers, "Accounts split their mutual covariance equally." At first glance this appears reasonable, but consider the following example.

Assume two accounts, (L) and (M). (M) has 100 times the losses of (L) for each event. Their total shared covariance is

$$\begin{aligned} 2 * \mathrm{Cov}(L, M) &= 2 * \Sigma_i \{ L_i * M_i * p_i * (1 - p_i) \} \\ &= 2 * \Sigma_i \{ L_i * 100L_i * p_i * (1 - p_i) \} \end{aligned} \quad [10.1]$$

The Shapley value would equally divide this total covariance between (L) and (M), even though their relative contributions to the total are clearly not equal. There is no question that (L) should be assessed *some* share of the covariance. The issue is whether there is a more equitable share than simply half.

We can develop a generalized covariance sharing (GCS) method which uses a weight $W_i^L(L, X)$ to determine (L)'s share of the mutual covariance between itself and account (X) for event i:

$$\mathrm{CovShare}_i^L (L, X) = W_i^L(L, X) * 2 * L_i * X_i * p_i * (1 - p_i) \quad [10.2]$$

Then (X)'s share of that mutual covariance would simply be

$$\mathrm{CovShare}_i^X (L, X) = [1 - W_i^L(L, X)] * 2 * L_i * X_i * p_i * (1 - p_i) \quad [10.3]$$

The total covariance share allocation for (L) over all events would be

$$\mathrm{CovShare}_{Tot}^L = \Sigma_Z \Sigma_i \{ \mathrm{CovShare}_i^L(L, Z) \}$$
$$\{ \Sigma_Z = \text{sum over every other account in the portfolio} \} \quad [10.4]$$

[15] Mr. Kreps [3] also incorporates covariance in his "Reluctance" R (p. 198), which has the formula R = [yz/(1+y)]/(2SC + σ)/(S' + S), where C is the correlation of the contract with the existing book. The Risk Load is then equal to Rσ.

The Shapley method is a generalized covariance sharing method with $W_i^L(L, X) = 50\%$ for all (L), (X), and i.

Returning to the example with (L) and (M), we can develop an example of a weighting scheme which assigns the shared covariance by event to each in proportion to their loss for that event. $W_i^L(L, M)$, account (L)'s share of the mutual covariance between itself and account (M) for event i, equals

$$W_i^L(L, M) = [L_i / [L_i + M_i]] = [L_i / [L_i + 100L_i]] = (1 / 101) \quad [10.5]$$

= roughly 1% of their mutual covariance for event i

We will call this the "Covariance Share" (CS) method.

(11) Applying the Shapley and CS Methods to the Example

Now we will see how the Shapley and CS methods perform in our 2 Account example for both Build-up and Renewal.

(11.1) Portfolio Build-up

Table C shows the Build-up of accounts (X) and (Y) from Section 5, but for the Shapley and CS methods. Here is a summary of the pertinent values from Table C for the Shapley value:

Table 11.1

Building Up (X) & (Y): Shapley Value	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Variance	19,619,900	1,828,509	21,448,409	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,353.02	$126.10	$1,479.11	$1,579.14

and for the Covariance Share:

Table 11.2

Building Up (X) & (Y): Covariance Share	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Variance	19,619,900	950,658	20,570,558	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,353.02	$65.56	$1,418.57	$1,579.14

Both Shapley and CS produce the same risk load for (X) as the MV method on build-up - $1,353.02. This is because there is no covariance to share - (X) is the entire portfolio at this point. However, let's compare the results of the three variance-based methods *for account* (Y):

Table 11.3

Comparison of Build-up Risk Loads for Account (Y)	
Marginal Variance (MV) - Section 5.2	$226.13
Shapley Value	$126.10
Difference from MV	***$100.03***
Covariance Share (CS)	$65.56
Difference from MV	***$160.57***

Compared to MV, which charges account (Y) for the full increase in variance (Var [Y] + 2* Cov [X, Y]), the Shapley method only charges (Y) for Var [Y] + Cov [X, Y]. The same can be said for the CS method, although the share of the mutual covariance depends on each account's relative contribution by event, weighted and summed over all events. Let's see what happens to that ***difference from MV*** upon renewal.

(11.3) Renewal

Table D shows the renewal of (X) and (Y) for the Shapley and CS methods. Here is a summary of pertinent values from Table D for the Shapley method:

Table 11.4

Renewing (X) & (Y): ***Shapley Value***	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)
(1) Change in Variance	21,070,450	1,828,509	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,453.05	$126.10	$1,579.14	$1,579.14
(4) Build-up Risk Load	$1,353.02	$126.10	$1,479.11	$1,579.14
(5) Difference	$100.03	$0	$100.03	$0

and for the Covariance Share method:

Table 11.5

Renewing (X) & (Y): ***Covariance Share***	Account (X)	Account (Y)	Account (X) + Account (Y)	Account (X + Y)

(1) Change in Variance	21,948,301	950,658	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,513.59	$65.56	$1,579.14	$1,579.14
(4) Build-up Risk Load	$1,353.02	$65.56	$1,418.57	$1,579.14
(5) Difference	$160.57	$0	$160.57	$0

With both the Shapley and CS methods, the sum of the risk loads for Account (X) and Account (Y) equals the risk load for Account (X + Y), namely $1,579.14. This means we have two **renewal additive** methods, which also means they are legitimate imputations.

To see what happened to ***difference from MV***, compare the risk loads calculated at renewal for (X) with those at build-up:

Table 11.6

Build-up vs Renewal Risk Loads for Account (X)	**Shapley**	**Cov Share**
Renewal	$1,453.05	$1,513.59
Build-up	$1,353.02	$1,353.02
Additional Renewal Risk Load over Build-up	***$100.03***	***$160.57***
Difference from MV	***$100.03***	***$160.57***

The difference from MV during build-up is simply the portion of (X)'s risk load attributable to its share of covariance with (Y). It was missed during build-up because it was unknown -- account (Y) had not been written.

(12) Conclusion

These new approaches address the concerns with renewal additivity, and point out the issue of covariance sharing between accounts. Perhaps the ideal solution might involve using a marginal method for the pricing of new accounts, and a renewal additive method for renewals. Any number of variations are possible, as long as one avoids double-counting the covariance.

It is hoped that this paper has also set the stage for further discussion of order dependency. This is a complex issue which was only touched on here, but which moves more to the forefront as advances in computer technology and modeling make ever finer levels of analysis possible.

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Table (A) Build a Portfolio of 2 Risks

Event i	P(i)	1-P(i)	Loss for Risk (X)	(Y)	(X + Y)
1	2.0%	98.0%	25,000	200	25,200
2	1.0%	99.0%	15,000	500	15,500
3	3.0%	97.0%	10,000	3,000	13,000
4	3.0%	97.0%	8,000	1,000	9,000
5	1.0%	99.0%	5,000	2,000	7,000
6	2.0%	98.0%	2,500	1,500	4,000

E[L]	1,290	179	1,469
Var[L]	19,619,900	377,959	22,898,959
Std Dev[L]	4,429	615	4,785

Covar	(X)	(Y)
(X)	19,619,900	1,450,550
(Y)	1,450,550	377,959

		(X)	(Y)	(X)+(Y)
Change in Std Deviation		4,171	356	4,526
Risk Load (Std Dev)		1,376.27	117.43	1,493.70
0.33	*Risk Load (A)*	1,461.71	117.43	1,579.14
	Difference	(85.45)		(85.45)

Change in Variance		22,521,000	3,279,059	25,800,059
Risk Load (Variance)		1,553.08	226.13	1,779.21
0.000069	*Risk Load (A)*	1,353.02	226.13	1,579.14
	Difference	200.06		200.06

Table (B) Renew the Portfolio of 2 Risks

Event i	P(i)	1-P(i)	Loss for Risk (X)	(Y)	(X + Y)
1	2.0%	98.0%	25,000	200	25,200
2	1.0%	99.0%	15,000	500	15,500
3	3.0%	97.0%	10,000	3,000	13,000
4	3.0%	97.0%	8,000	1,000	9,000
5	1.0%	99.0%	5,000	2,000	7,000
6	2.0%	98.0%	2,500	1,500	4,000

E[L]	1,290	179	1,469
Var[L]	19,619,900	377,959	22,898,959
Std Dev[L]	4,429	615	4,785

Covar	(X)	(Y)
(X)	19,619,900	1,450,550
(Y)	1,450,550	377,959

		(X)	(Y)	(X)+(Y)
Change in Std Deviation		4,171	356	4,526
Risk Load (Std Dev)		1,376.27	117.43	1,493.70
0.33	*Risk Load (A)*	1,461.71	117.43	1,579.14
	Difference	(85.45)		(85.45)

Change in Variance		22,521,000	3,279,059	25,800,059
Risk Load (Variance)		1,553.08	226.13	1,779.21
0.000069	*Risk Load (A)*	1,353.02	226.13	1,579.14
	Difference	200.06		200.06

Table (C) Build a Portfolio of 2 Risks - Alternatives

			Covariance Share $(Y)	
1	2.0%	98.0%	9,920,635	79,365
2	1.0%	99.0%	14,516,129	483,871
3	3.0%	97.0%	46,153,846	13,846,154
4	3.0%	97.0%	14,222,222	1,777,778
5	1.0%	99.0%	14,285,714	5,714,286
6	2.0%	98.0%	4,687,500	2,812,500

			Total
	2,328,401	572,699	2,901,100

Chg in Variance		(X)	(Y)
If added 1st		19,619,900	377,959
If added 2nd	after 1		3,279,059
	after 2	22,521,000	
Average *(Shapley Value)*		21,070,450	1,828,509

Shapley Value		21,070,450	1,828,509	22,898,959
Risk Load (Shapley)		**1,453.05**	126.10	1,579.14
0.000069	*Risk Load (C)*	1,353.02	126.10	
	Difference	**100.03**	*= Deferred Risk Load from (C)*	

Covariance Share		21,948,301	950,658	22,898,959
Risk Load (Cov Share)		**1,513.59**	65.56	1,579.14
0.000069	*Risk Load (C)*	1,353.02	65.56	
	Difference	**160.57**	*= Deferred Risk Load from (C)*	

Table (D) Renew the Portfolio of 2 Risks - Alternatives

			Covariance Share $	
Event i	P(i)	1-P(i)	(X)	(Y)
1	2.0%	98.0%	9,920,635	79,365
2	1.0%	99.0%	14,516,129	483,871
3	3.0%	97.0%	46,153,846	13,846,154
4	3.0%	97.0%	14,222,222	1,777,778
5	1.0%	99.0%	14,285,714	5,714,286
6	2.0%	98.0%	4,687,500	2,812,500

			Total
	2,328,401	572,699	2,901,100

Chg in Variance		(X)	(Y)
If added 1st		19,619,900	377,959
If added 2nd	after 1		3,279,059
	after 2	22,521,000	
Average *(Shapley Value)*		21,070,450	1,828,509

Shapley Value		21,070,450	1,828,509	22,898,959
Risk Load (Shapley)		**1,453.05**	126.10	1,579.14
0.000069	*Risk Load (C)*	1,353.02	126.10	
	Difference	**100.03**	*= Deferred Risk Load from (C)*	

Covariance Share		21,948,301	950,658	22,898,959
Risk Load (Cov Share)		**1,513.59**	65.56	1,579.14
0.000069	*Risk Load (C)*	1,353.02	65.56	
	Difference	**160.57**	*= Deferred Risk Load from (C)*	